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CONVEX AND STARLIKE FUNCTIONS

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1. Introduction. Let S denote the class of functions which are regular and univalent in the unit disc $E = \{z: |z| < 1\}$ and normalized such that $f(0) = f'(0)-1 = 0$. A function f belonging to S is said to be *starlike of order* α, $0 \leqslant \alpha < 1$, if and only if $\operatorname{Re}(zf'(z)/f(z)) > \alpha$, $z \in E$, and we denote by $\mathrm{St}(\alpha)$ the class of all such functions. $\mathrm{St} = \mathrm{St}(0)$ will be referred to as the class of starlike functions in E. The class $K(\alpha)$, $0 \leqslant \alpha < 1$, of convex functions of order α, consists of all those elements $f \in S$ which satisfy the condition $\operatorname{Re}(1+zf''(z)/f'(z)) > \alpha$, $z \in E$ and $K = K(0)$ will be referred to as the class of convex functions in E. It is well known that $K \subset \mathrm{St}(1/2)$.

Strohhäcker [8] in 1933 proved that if $f \in K$, then $\operatorname{Re}(f(z)/z) > 1/2$ in E. His result was generalized by Sheil-Small [7] who showed that if

$$f \in K, \qquad f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

then for each integer $n \geqslant 1$, we have $\operatorname{Re}(f(z)/s_n(z,f)) > 1/2$, $z \in E$, where

$$s_n(z,f) = z + \sum_{k=2}^{n} a_k z^k$$

is the nth partial sum of $f(z)$. Ruscheweyh and Sheil-Small [6] further generalized this latter result by showing that $\operatorname{Re}(f(z)/s_n(z,f)) > 1/2$, $z \in E$, holds even if $f \in \mathrm{St}(1/2)$.

In the present paper, making use of two powerful theorems of Ruscheweyh and Sheil-Small [6], we obtain some results which relate convex and starlike functions of order 1/2 with their partial sums. Some well known results follow as particular cases from our results. We also give alternative simple proofs of three well known results one of which is the Ruscheweyh–Sheil-Small theorem mentioned earlier and the other two pertain to convex functions.

2. Definitions and lemmas. The *Hadamard product* or *convolution* of two power series $g(z)=\sum_{k=0}^{\infty} a_k z^k$ and $h(z)=\sum_{k=0}^{\infty} b_k z^k$ is defined as the power series

$$(g*h)(z)=\sum_{k=0}^{\infty} a_k b_k z^k.$$

The *n-th de la Vallée Poussin mean* of an analytic function $f(z) = \sum_{k=1}^{\infty} a_k z^k$ is the polynomial defined by

$$v_n(z,f)=\frac{(n!)^2}{(2n)!}\sum_{k=1}^{n}\frac{(2n)!}{(n-k)!(n+k)!}a_k z^k$$
$$=\frac{n}{n+1}a_1 z+\frac{n(n-1)}{(n+1)(n+2)}a_2 z^2+\ \dots\ +\frac{n(n-1)\dots 2\cdot 1}{(n+1)(n+2)\dots(2n)}a_n z^n.$$

The *n-th Cesàro mean* of the first order of an analytic function $f(z) = \sum_{k=1}^{\infty} a_k z^k$ is defined by

$$\sigma_n(z,f)=\frac{1}{n}(s_1(z,f)+s_2(z,f)+\ \dots\ +s_n(z,f)),$$

where $s_m(z,f)$ is the mth partial sum of $f(z)$.

If g is regular in E, h is regular and univalent in E, $g(0)=h(0)$, then by the notation $g \prec h$ (*g is subordinate to h*) in E, we shall mean that $g(E)\subset h(E)$.

The following two lemmas are immediate consequences of Remark (2.5) and Lemmas (2.7) and (3.5) of [6], due to Ruscheweyh and Sheil-Small.

LEMMA 1. *Let φ be convex and g starlike in E. Then $(\varphi * gF)/(\varphi * g)$ takes values in the convex hull of $F(E)$ for every function F analytic in E.*

LEMMA 2. *If φ and ψ are starlike of order 1/2 in E, then $(\varphi * \psi F)/(\varphi * \psi)$ takes values in the convex hull of $F(E)$ for every function F analytic in E.*

3. Theorems and their proofs.

THEOREM 1. *If $f\in K$, $f(z)=z+\sum_{k=2}^{\infty} a_k z^k$, then for each integer $n\geqslant 1$, we have*

$$\operatorname{Re}\frac{f(z)-f(-z)}{s_n(z,f)-s_n(-z,f)}>\frac{1}{2}\quad (z\in E).$$

Proof. Clearly it is enough to prove the result when n is an odd integer. So let us assume that $n=2m+1$, where $m\geqslant 0$ is any integer. Since $f(z)=z+a_2 z^2+\ \dots$ belongs to K and $g(z)=z(1-z^2)^{-1}$ is in St, it follows from Lemma 1 that for all z in E, the function w, defined by

$$w(z) = \frac{f(z) * \frac{z}{1-z^2}(1-z^{2m+2})}{f(z) * \frac{z}{1-z^2}},$$

takes values in the convex hull of $F(E)$, where $F(z) = 1-z^{2m+2}$. However, it is readily seen that

$$w(z) = \frac{s_{2m+1}(z,f)-s_{2m+1}(-z,f)}{f(z)-f(-z)} = \frac{s_n(z,f)-s_n(-z,f)}{f(z)-f(-z)}.$$

We, therefore, conclude that

$$\left|\frac{s_n(z,f)-s_n(-z,f)}{f(z)-f(-z)} - 1\right| < 1 \quad (z \in E),$$

from which the assertion of our theorem follows at once.

It is well known that if f belongs to K, then the function h, defined by

$$h(z) = \frac{f(z)-f(-z)}{2},$$

is in St. This fact and Theorem 1 together lead us to the following

COROLLARY 1. *If f belongs to K, $f(z) = z+\sum_{k=2}^{\infty} a_k z^k$, then for each integer $n \geqslant 0$, the polynomial*

$$P_n(z) = \sum_{k=0}^{n} \frac{a_{2k+1}}{2k+1} z^{2k+1}, \quad a_1 = 1,$$

is close-to-convex [2] *and hence univalent in E.*

Proof. We have

$$P_n(z) = \int_0^z \frac{s_{2n+1}(t,f)-s_{2n+1}(-t,f)}{2t}\,dt,$$

from which we obtain

$$\mathrm{Re}\left[\frac{2zP_n'(z)}{f(z)-f(-z)}\right] = \mathrm{Re}\frac{s_{2n+1}(z,f)-s_{2n+1}(-z,f)}{f(z)-f(-z)} > 0 \quad (z \in E),$$

This shows that $P_n(z)$ is close-to-convex with respect to the starlike function h, $h(z) = (f(z)-f(-z))/2$.

If in Lemma 1 we take $g(z) = z/(1-z^2)$, $F(z) = (1+z)(1-z)^{-1}$ we conclude that for every f belonging to K,

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0, \qquad z \in E.$$

This fact and Corollary 1 yield:

COROLLARY 2. *If f belongs to K, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, and $P_n(z)$ is defined as in Corollary* 1, *then the function $L_n(z)$, defined by*

$$L_n(z) = \tfrac{1}{2}[f(z) + P_n(z)]$$

is close-to-convex and hence univalent in E for each integer $n \geqslant 0$.

We now give a very simple alternative proof of the theorem of Ruscheweyh and Sheil-Small mentioned in Section 1, namely

THEOREM 2. *If f belongs to* St(1/2), *$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then for every integer $n \geqslant 1$, we have*

$$\operatorname{Re}\frac{f(z)}{s_n(z, f)} > \frac{1}{2} \qquad (z \in E),$$

where $s_n(z, f)$ is the n-th partial sum of $f(z)$.

Proof. Taking $\varphi(z) = f(z)$, $\psi(z) = z/(1-z)$, and $F(z) = 1 - z^n$ in Lemma 2, we infer that the function

$$p(z) = \frac{f(z) * \dfrac{z}{1-z}(1-z^n)}{f(z) * \dfrac{z}{1-z}} = \frac{s_n(z, f)}{f(z)}$$

takes values in the convex hull of $F(E)$ and as such satisfies

$$\left|\frac{s_n(z, f)}{f(z)} - 1\right| < 1 \qquad (z \in E).$$

The conclusion of Theorem 2 is now clear.

THEOREM 3. *If f belongs to* St(1/2), *$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

(a) $$\operatorname{Re}\frac{f(z)}{v_1(z, f)} > 1 \qquad (z \in E),$$

and

(b) $$\operatorname{Re}\frac{f(z)}{v_n(z, f)} > 0 \qquad (z \in E)$$

for each integer $n > 1$, *where* $v_n(z, f)$ *is the de la Vallée Poussin mean of order* n *of the function* f.

Proof. Since $v_1(z, f) = z/2$, case (a) follows from Theorem 2. To prove (b) we choose in Lemma 2 the functions φ, ψ and F as follows:

$$\varphi(z) = f(z), \quad \psi(z) = \frac{z}{1-z}$$

and

$$F(z) = F_n(z) = \frac{3n}{(n+1)(n+2)}(1-z)+$$

$$+\frac{5n(n-1)}{(n+1)(n+2)(n+3)}(1-z^2)+ \ldots +\frac{(2n+1)n(n-1)(n-2)\ldots 2\cdot 1}{(n+1)(n+2)\ldots(2n)(2n+1)}(1-z^n).$$

Clearly $\operatorname{Re} F(z) > 0$ in E and therefore we conclude that $\operatorname{Re} p(z) > 0$, $z \in E$, where

$$p(z) = \frac{f(z) * \dfrac{z}{1-z} F(z)}{f(z) * \dfrac{z}{1-z}}.$$

It is now easily verified that in fact $p(z)$ equals $v_n(z, f)/f(z)$. This completes the proof of Theorem 3.

COROLLARY 3. *Let* f *belong to* $\mathrm{St}(1/2)$ *and for each integer* $n \geqslant 1$, *define*

$$g_n(z) = \frac{n+1}{2n+1}[f(z) + V_n(z, f)]$$

where

$$V_n(z, f) = \int_0^z \left((v_n(t, f))/t\right) dt;$$

then g_n *is close-to-convex* (*of order* $(n+1)/2(2n+1)$), *and hence a member of* S.

THEOREM 4. *If* f *belongs to* $\mathrm{St}(1/2)$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *then for each integer* $n \geqslant 1$, *we have*

$$\operatorname{Re}(1 \pm a_{n+1} z^n + a_{2n+1} z^{2n} \pm a_{3n+1} z^{3n} + \ldots) > 1/2 \quad (z \in E).$$

Proof. In Lemma 2, letting $\varphi(z) = f(z)$, $\psi(z) = z$ and $F(z) = (1 \mp z^n)^{-1}$, we conclude that for all $z \in E$ the function

$$\frac{f(z) * z(1 \mp z^n)^{-1}}{f(z) * z}$$

takes values in the convex hull of $F(E)$. Since $\operatorname{Re} F(z) > 1/2$ in E, the desired result follows.

THEOREM 4, in particular, implies Strohhäcker's theorem and generalizes, for $n = 1$, the result of Ruscheweyh and Sheil-Small mentioned in Section 1.

THEOREM 5. *If $f \in K$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then for each integer $n \geq 0$, we have*

$$\operatorname{Re} \frac{zf'(z)}{(n+1)f(z) - n\sigma_n(z,f)} > \frac{1}{2} \quad (z \in E),$$

where $\sigma_n(z,f)$ is the n-th Cesàro mean of $f(z)$ of the first order.

Proof. In Lemma 1, letting $\varphi(z) = f(z)$, $g(z) = z/(1-z)^2$ and $F(z) = 1 - z^{n+1}$, we conclude that for all $z \in E$ the function

$$q(z) = \frac{f(z) * \dfrac{z}{(1-z)^2}(1-z^{n+1})}{f(z) * \dfrac{z}{(1-z)^2}}$$

$$= \frac{f(z) * \left[(n+1)\dfrac{z}{1-z} - \{nz + (n-1)z^2 + \ldots + 2z^{n-1} + z^n\} \right]}{zf'(z)}$$

takes values in the convex hull of $F(E)$, and therefore the inequality

$$\left| \frac{(n+1)f(z) - n\sigma_n(z,f)}{zf'(z)} - 1 \right| < 1$$

holds in E. The assertion of our theorem is now clear.

For $n = 0$, Theorem 5 yields the known result:

COROLLARY 4. $K \subset \mathrm{St}(1/2)$.

COROLLARY 5. *If f belongs to K, then for each integer $n \geq 0$, the function F, defined by*

$$F(z) = \int_0^z \frac{(n+1)f(t) - n\sigma_n(t,f)}{t} dt,$$

is close-to-convex and hence univalent in E.

In the following two theorems we give alternative proofs of two known results about convex functions, first one of which was established by MacGregor [4] and the second one forms a part of a more general result due to Brickman *et al.* [1].

THEOREM 6. *Let f belong to K and define*

$$\Delta_f(z_1, z_2) = \begin{cases} \dfrac{f(z_1)-f(z_2)}{z_1-z_2} & (z_1, z_2 \in E,\ z_1 \neq z_2), \\ f'(z) & (z_1 = z_2 = z). \end{cases}$$

Then $|\Delta_f(z_1, z_2)| > 1/4$.

Proof. It is readily verified that the function

$$g(z, \zeta) = \frac{z}{(1-z)(1-\zeta z)} \quad (|\zeta| \leqslant 1)$$

is in St. Hence, in view of Lemma 1, we conclude that the function

$$\frac{f(z) * (1-\zeta)\dfrac{z}{(1-z)(1-\zeta z)}[(1-z)(1-\zeta z)]}{f(z) * (1-\zeta)\dfrac{z}{(1-z)(1-\zeta z)}} = \left[\frac{f(z) * (z-\zeta z)}{f(z) * \left[\dfrac{z}{1-z} - \dfrac{z\zeta}{1-\zeta z}\right]}\right] = \frac{z-\zeta z}{f(z)-f(\zeta z)}$$

takes values in the convex hull of $F(E)$, where

$$F(z) = F(z, \zeta) = (1-z)(1-\zeta z).$$

Since the convex hull of $F(E)$ is contained in the disc with centre at the origin and radius $= \sup\limits_{|z|<1, |\zeta| \leqslant 1} |F(z, \zeta)|$, it follows that

$$\left|\frac{z-\zeta z}{f(z)-f(\zeta z)}\right| < 4$$

and Theorem 6 is established.

THEOREM 7. *If f belongs to $K(1/2)$, $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$, then*

$$\frac{f(z)}{z} \prec -\frac{1}{z}\log(1-z) \quad (z \in E),$$

and hence

$$\operatorname{Re}\frac{f(z)}{z} > \log 2 \quad (z \in E).$$

Proof. Define g by

$$f(z) = \int_0^z \frac{g(t)}{t}\,dt,$$

then g belongs to St(1/2) and $g(z) = z + \sum_{k=2}^{\infty} k a_k z^k$. Let

$$F(z) = -\frac{2}{z}\log(1-z) - 1 \qquad (z \in E).$$

It is known [5] that F is univalent in E and maps E onto a convex domain and $\operatorname{Re} F(z) > 2\log 2 - 1$ $(z \in E)$. From Lemma 2 we then conclude that the function

$$w(z) = \frac{g(z) * zF(z)}{g(z) * z} = \frac{g(z) * [-2\log(1-z) - z]}{z} = \frac{2f(z)}{z} - 1$$

takes values in the convex hull of $F(E)$. Since the convex hull of $F(E)$ is $F(E)$ itself, we have

$$\frac{2f(z)}{z} - 1 \prec -\frac{2}{z}\log(1-z) - 1 \qquad (z \in E)$$

or

$$\frac{f(z)}{z} \prec -\frac{1}{z}\log(1-z) \qquad (z \in E).$$

This completes the proof of Theorem 7.

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