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A CHARACTERIZATION OF SPHERES AMONG CONVEX 3-BODIES

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Suppose K is a compact subset of E^3 having a surface area s which has interior points. Denote by $p(\boldsymbol{v})$ the area of the two-dimensional projection of K in the vector direction $\boldsymbol{v}$. Let

$$p = \inf_{\boldsymbol{v}} p(\boldsymbol{v}).$$

Now, let $q(\boldsymbol{v})$ be the area of that planar cross-section of K normal to the direction $\boldsymbol{v}$ which bisects the surface area s. (Alternatively, the cross-section can be chosen in other ways, e. g., to bisect the volume of K.) For any direction $\boldsymbol{v}$, $p(\boldsymbol{v}) \geqslant q(\boldsymbol{v})$. Let

$$q = \inf_{\boldsymbol{v}} q(\boldsymbol{v}),$$

so $p \geqslant q$. My conjecture is that *always $s \geqslant 4q$, and $s = 4q$ iff K is a sphere.*

If K is also *convex*, this conjecture is true. The surface area is then given by the surface integral

$$s = \frac{1}{\pi} \int_U p(\boldsymbol{v}) d\boldsymbol{v},$$

where U is the unit sphere ([2], p. 67). Since $p(\boldsymbol{v})$ is now a continuous function of $\boldsymbol{v}$ ([2], p. 40 and 45),

$$p = \min_{\boldsymbol{v}} p(\boldsymbol{v})$$

exists; then

$$s \geqslant \frac{1}{\pi} p \int_U d\boldsymbol{v} = \frac{1}{\pi} p \cdot 4\pi = 4p.$$

Hence also $s \geqslant 4q$, and $s = 4p$ iff K has constant two-dimensional projection in every direction. Now, if $s = 4q$, then $q(\boldsymbol{v}) \equiv p(\boldsymbol{v}) \equiv p = q$. Thus, the two-dimensional projection of K is constant in this case, and the intersection of K with generators of each fixed circumscribed cylinder contains a planar cross-section of the boundary of K. A theorem due to Marchaud ([4], p. 283), which is a sharpened version of a theorem of Blaschke ([1], p. 157-159; see also [2], p. 142), states that K must then be an ellipsoid. Having constant two-dimensional projection, K must be a sphere.

The two-dimensional version of this problem has been solved in a variety of settings, without the convexity assumption. For details, consult the survey article [3] and the papers cited therein.

REFERENCES

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[4] A. Marchaud, *Un théorème sur les corps convexes*, Annales Scientifiques de l'École Normale Supérieure, 3-ème sér., 76 (1959), p. 283-304.

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