COLLOQUIUM MATHEMATICUM

VOL. XLVI 1982 FASC. 2

JOINT CONTINUITY OF SEQUENTIALLY CONTINUOUS MAPS

BY

D. HELMER (CONSTANCE)

Let $f\colon X\times Y\to Z$ be a separately continuous and sequentially continuous map, where Z is completely regular. The purpose of this note is to prove the following

THEOREM. *Suppose X is a G_δ-subset or a Baire subset of some locally compact group and Y is of the same sort. Then f is continuous.*

The proof will be based on the following proposition that allows reduction to a separable situation whenever joint continuity of a separately continuous map on (countably) compact sets is to be derived:

PROPOSITION. *Suppose $X\times Y$ has a dense subset D such that each sequence in D clusters in $X\times Y$. Then f is continuous provided it is continuous on countable subspaces of $X\times Y$.*

Proof. We may assume that $Z=\boldsymbol{R}$ and then that f is bounded. Let $X^*=\{x^*\mid x\in X\}$, where x^* denotes the map $y\mapsto f(x,y)\colon Y\to\boldsymbol{R}$. It follows from Théorème 6 in [4] that – with respect to the supremum norm topology on $C(Y)$ – the weak closure $\bar{X}^*$ is weakly compact. For rea-ons of symmetry, $\bar{Y}^*$ is weakly compact in the Banach space $C(X)$. Consequently ([7], p. 313), $X^*\times Y^*$ is hereditarily sequential. As f factors through the continuous map

$$\pi\colon X\times Y\to X^*\times Y^*\quad((x,y)\mapsto(x^*,y^*))$$

and $\pi(D)$ is dense in $X^*\times Y^*$, it suffices to show that if $(x_n^*,y_n^*)\to(x^*,y^*)$ is a convergent sequence in $X^*\times Y^*$ with $(x_n,y_n)\in D$, then $f(x_n,y_n)$ clusters at $f(x,y)$. However, some subnet $(x_{n_\alpha},y_{n_\alpha})$ of (x_n,y_n) converges in $X\times Y$, say to (u,v). Since f is continuous on $(\{x_{n_\alpha}\}\cup\{u\})\times(\{y_{n_\alpha}\}\cup\{v\})$, we obtain $f(x_{n_\alpha},y_{n_\alpha})\to f(u,v)$ in $\boldsymbol{R}$. But $x_{n_\alpha}^*\to u^*$ in X^*, whence $u^*=x^*$. Similarly, $v^*=y^*$. Thus, $f(u,v)=f(x,y)$.

Proof of the Theorem. Let G, H be locally compact groups containing X, Y as subspaces, respectively. Let then G_0 stand for the connected component of the identity in G. Fix a compact open subgroup L in G/G_0 ([5], p. 62). Since G/G_0 is zero-dimensional, the quotient map

$G \to G/G_0$ admits a cross-section ([8], Theorem 8). Consequently, G is homeomorphic to $G_0 \times G/G_0$. Now G_0, in turn, is homeomorphic to the product $\boldsymbol{R}^n \times K$ of a Euclidean space and any maximal compact subgroup K ([6], p. 549). So, there is no loss of generality in assuming that one is dealing with the group $G_1 = \boldsymbol{R}^n \times K \times G/G_0$ instead of G.

Let $A \times B \subseteq X \times Y$ be compact. First, we show that A is contained in a dyadic subspace E of X that has a countable neighborhood base in X. To this end, choose a compact subspace C of $\boldsymbol{R}^n$ and finitely many $\gamma_1, \ldots, \gamma_m \in G/G_0$ such that A is contained in

$$D = \bigcup_{i=1}^{m} C \times K \times \gamma_i L.$$

By Kuzminov's theorem ([9], 7.6), D is dyadic. Assume now X is a Baire subset of G_1, i.e., a member of the σ-algebra generated by the zero-sets of G_1. Then $X \cap D$ is a Baire subset of D. Consequently ([2], p. 69), there exists a continuous map $g\colon D \to M$ onto some metrizable space such that $X \cap D = g^{-1}(g(X \cap D))$. Since $g(A)$ is compact in M, it follows that $E = g^{-1}(g(A))$ is a compact G_δ-subspace of D and as such is dyadic as well ([3], p. 300). As E has a countable neighborhood base in D, and D, in turn, has a countable neighborhood base in G_1, E has a countable neighborhood base in G_1, hence in X. On the other hand, assume that

$$X = \bigcap_{j=1}^{\infty} Q_j$$

with Q_j open in G_1. For every j, choose a compact G_δ-subset E_j of G_1 in such a way that $A \subseteq E_j \subseteq Q_j$. Then

$$E = D \cap \bigcap_{j=1}^{\infty} E_j$$

is as desired.

For reasons of symmetry, B lies in a dyadic subspace F of Y that has a countable neighborhood base in Y. In particular, $X \times Y$ is a k-space ([1], p. 37). Consequently, it is good enough to verify continuity of f on $E \times F$. To this end, fix Cantor spaces $\mathfrak{D}^{\mathfrak{m}}$ and $\mathfrak{D}^{\mathfrak{n}}$ and continuous surjections $\alpha\colon \mathfrak{D}^{\mathfrak{m}} \to E$ and $\beta\colon \mathfrak{D}^{\mathfrak{n}} \to F$. Let $(s, t) \in \mathfrak{D}^{\mathfrak{m}} \times \mathfrak{D}^{\mathfrak{n}}$. If S is the subspace of $\mathfrak{D}^{\mathfrak{m}}$ consisting of all $\mathfrak{m}$-tuples differing from s only at a countable number of places and if $T \subseteq \mathfrak{D}^{\mathfrak{n}}$ is analogous for t, then $\alpha(S) \times \beta(T)$ is dense in $E \times F$. By regularity of Z, in order that $E \times F \to Z$ be continuous at $(\alpha(s), \beta(t))$ it suffices to show continuity of $\alpha(S) \times \beta(T) \to Z$ at $(\alpha(s), \beta(t))$. Since S and T are sequentially compact, by the Proposition it is enough to have $\alpha(U) \times \beta(V) \to Z$ continuous for any given countable subset $U \times V$ of $S \times T$. However, $\alpha(U) \times \beta(V)$ is metrizable, since $\overline{U} \times \overline{V}$ is compact and metrizable.

Remarks. 1. Let $\mathscr{S}$ denote the class of all completely regular Hausdorff spaces S of countable type (cf. [1], p. 37) that have the property that every separable compactum of S is contained in a dyadic subspace of S. Modifying somewhat the above proof, one sees that the Theorem remains valid if X and Y are Baire or G_δ-subsets of members S and T of $\mathscr{S}$, respectively. $\mathscr{S}$ comprises all coset spaces of locally compact groups. It also enjoys much better permanence properties than the class of locally compact groups, e.g., with respect to the formation of countable products, G_δ-subspaces, and Baire subspaces.

2. The separate continuity hypothesis is essential in the Theorem. Indeed, in [10], p. 456, Varopoulos has constructed a discontinuous, sequentially continuous character f on some compact Abelian group X. If Y is a singleton, then f may be considered as a map on $X \times Y$.

REFERENCES

[1] A. V. Arhangel'skiĭ, *Bicompact sets and the topology of spaces*, Transactions of the Moscow Mathematical Society (1965), p. 1-62.

[2] W. W. Comfort and S. Negrepontis, *Continuous pseudometrics*, New York 1975.

[3] R. Engelking, *Cartesian products and dyadic spaces*, Fundamenta Mathematicae 57 (1965), p. 287-304.

[4] A. Grothendieck, *Critères de compacité dans les espaces fonctionnels généraux*, American Journal of Mathematics 74 (1952), p. 168-186.

[5] E. Hewitt and K. A. Ross, *Abstract harmonic analysis. I*, Berlin 1963.

[6] K. Iwasawa, *On some types of topological groups*, Annals of Mathematics 50 (1949), p. 507-558.

[7] G. Köthe, *Topological vector spaces. I*, New York 1969.

[8] P. S. Mostert, *Sections in principal fiber spaces*, Duke Mathematical Journal 23 (1956), p. 57-72.

[9] A. Pełczyński, *Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions*, Dissertationes Mathematicae 58, Warszawa 1968.

[10] N. Th. Varopoulos, *A theorem on the continuity of homomorphisms of locally compact groups*, Proceedings of the Cambridge Philosophical Society 60 (1964), p. 449-463.

Reçu par la Rédaction le 20. 7. 1978;
en version modifiée le 13. 5. 1980