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MONOTONE SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS

BY

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Throughout the whole paper we suppose n to be an odd integer. The aim of this paper is to prove the existence of a solution of the differential equation

$$y^{(n)}+B(x,y,y',\ldots,y^{(n-1)})y = 0 \tag{E}$$

which has the properties:

$$\text{(V)}\quad \begin{cases} (-1)^i y^{(i)}(x) > 0 \text{ or } (-1)^{i+1}y^{(i)}(x) > 0, & i = 0,1,\ldots,n-1, \\ \lim\limits_{x\to\infty} y^{(i)}(x) = 0, \quad i = 1,2,\ldots,n-1, \\ \lim\limits_{x\to\infty} y(x) \text{ exists and is not zero.} \end{cases}$$

I. We shall be concerned at first with the linear differential equation

$$y^{(n)}+Q(x)y = 0. \tag{1}$$

We suppose that $Q(x)$ is a non-negative continuous function on the interval (a, ∞), $-\infty \leqslant a$, which does not equal zero in any sub-interval of (a, ∞). Under these conditions we prove some lemmas and theorems.

LEMMA 1. *The solution $y(x)$ of* (1), *which is determined by the initial conditions*

$$\begin{aligned} y^{(s)}(x_0) &= 0, \quad s = 0,1,\ldots,n-2, \quad x_0 \in (a, \infty), \\ y^{(n-1)}(x_0) &\neq 0, \end{aligned} \tag{2}$$

has no zero less than x_0 and for $x < x_0$ it holds:

$$(-1)^i y^{(i)}(x) > 0 \text{ or } (-1)^{i+1}y^{(i)}(x) > 0, \quad i = 0,1,\ldots,n-1. \tag{3}$$

Proof. Let $t = x_0-x$. Then $Q(x_0-t) = Q^*(t) \geqslant 0$, $y(x) = y(x_0-t) = u(t)$ and

$$y^{(i)}(x) = (-1)^i \frac{d^i u(t)}{dt^i}, \quad i = 0,1,\ldots,n. \tag{4}$$

From (1) we get for $u(t)$ the equation

$$\frac{d^n u}{dt^n} - Q^*(t)u(t) = 0, \tag{5}$$

and from (2) the initial conditions

$$\begin{aligned} &\frac{d^i u(0)}{dt^i} = 0, \quad i = 0, 1, \ldots, n-2, \\ &\frac{d^{n-1}u(0)}{dt^{n-1}} \neq 0. \end{aligned} \tag{6}$$

Suppose $y^{(n-1)}(x_0) > 0$. Then $d^{n-1}u(0)/dt^{n-1} > 0$. Next from (5) and (6) we have

$$\frac{d^i u(t)}{dt^i} > 0 \quad \text{for} \quad t > 0, \ i = 0, 1, \ldots, n-1,$$

and, according to (4), $(-1)^i y^{(i)}(x) > 0$ for $x < x_0$, $i = 0, 1, \ldots, n-1$. For $y^{(n-1)}(x_0) < 0$ we get $(-1)^{i+1} y^{(i)}(x) > 0$, $i = 0, 1, \ldots, n-1$.

THEOREM 1. *Equation* (1) *has a solution* $y(x)$ *having the following properties*:

$$(-1)^i y^{(i)}(x) > 0 \ \textit{or} \ (-1)^{i+1} y^{(i)}(x) > 0, \quad i = 0, 1, \ldots, n-1, \tag{7}$$

for $x \in (a, \infty)$;

$$\lim_{x \to \infty} y^{(i)}(x) = 0, \quad i = 1, 2, \ldots, n-1; \tag{8}$$

$$\lim_{x \to \infty} y(x) \ \textit{exists and is finite}. \tag{9}$$

Proof. Let $x_0 \in (a, \infty)$ and let $\{x_k\}_{k=1}^{\infty}$ be a sequence of numbers such that $x_k \in (a, \infty)$, $x_k < x_{k+1}$, $\lim_{k \to \infty} x_k = \infty$. Let $y_k(x)$ be a solution of (1) which satisfies at x_k conditions (2) and let $y_k^{(n-1)}(x_k) > 0$,

$$\sum_{i=0}^{n-1} [y_k^{(i)}(x_0)]^2 = 1. \tag{10}$$

By the use of Lemma 1 we have for $x < x_k$

$$(-1)^s y_k^{(s)}(x) > 0, \quad s = 0, 1, \ldots, n-1. \tag{11}$$

Let now $z_j(x)$, $j = 0, 1, \ldots, n-1$, be a fundamental system of solutions of (1) determined by the initial conditions

$$z_j^{(i)}(x_0) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}, \quad i, j = 0, 1, \ldots, n-1. \tag{12}$$

Then we can express $y_k(x)$ as follows:

$$y_k(x) = \sum_{i=0}^{n-1} c_i^{(k)} z_i(x), \qquad y_k^{(i)}(x_0) = c_i^{(k)}. \tag{13}$$

Furthermore, condition (10) yields

$$\sum_{i=0}^{n-1} [c_i^{(k)}]^2 = 1. \tag{14}$$

Hence it follows that the sequences $\{c_i^{(k)}\}_{k=1}^{\infty}$, $i = 0, 1, \ldots, n-1$, are bounded. So it is possible to choose convergent subsequences. Let the sequences $\{c_i^{(k)}\}_{k=1}^{\infty}$, $i = 0, 1, \ldots, n-1$, be so chosen and let

$$\lim_{k\to\infty} c_i^{(k)} = a_i, \qquad i = 0, 1, \ldots, n-1. \tag{15}$$

Then, from (13), we have

$$\lim_{k\to\infty} y_k(x) = \lim_{k\to\infty} \sum_{i=0}^{n-1} c_i^{(k)} z_i(x) = \sum_{i=0}^{n-1} \lim_{k\to\infty} c_i^{(k)} z_i(x) \tag{16}$$

$$= \sum_{i=0}^{n-1} a_i z_i(x) = y(x).$$

The function $y(x)$ is a non-trivial solution of (1), as it follows from formula (14).

Now, from (11) and (13) we get

$$\lim_{k\to\infty} (-1)^s y_k^{(s)}(x) = (-1)^s y^{(s)}(x) \geqslant 0, \qquad s = 0, 1, \ldots, n-1, \tag{17}$$

for $x \in (a, \infty)$, and from (1)

$$y^{(n)}(x) \leqslant 0. \tag{18}$$

From the fact that $y(x)$ is a non-trivial solution of (1) and from (17) and (18) it is easy to prove that

$$(-1)^s y^{(s)}(x) > 0 \text{ for } x \in (a, \infty), \qquad i = 0, 1, \ldots, n-1,$$

$$\lim_{x\to\infty} y^{(s)}(x) = 0, \qquad s = 1, 2, \ldots, n-1,$$

$\lim_{x\to\infty} y(x)$ exists and is non-negative and finite.

In the case $y_k^{(n-1)}(x_k) < 0$ the same type of reasoning may be used to prove the existence of a solution of (1) having properties (7)-(9).

THEOREM 2. *Let $y(x)$ be a non-trivial solution of* (1) *having the properties*

$$\text{(B)} \qquad \begin{cases} \lim_{n\to\infty} y^{(i)}(x) = 0, \quad i = 1, 2, \ldots, n-1, \\ \lim_{x\to\infty} y(x) \text{ exists and is finite.} \end{cases}$$

Then $\lim_{x\to\infty} y(x) = 0$ *iff*

$$\int^{\infty} x^{n-1} Q(x)\, dx = \infty. \tag{19}$$

Proof. Suppose $y(x)$ is a non-trivial solution of (1) having properties (B) and let $\lim_{x\to\infty} y(x) = 0$ and $\int^{\infty} x^{n-1} Q(x)\, dx < \infty$. Then there exists a number c such that

$$\frac{1}{(n-1)!} \int_c^{\infty} (t-c)^{n-1} Q(t)\, dt < 1. \tag{20}$$

By integrating (1) and using the properties of $y(x)$ we get

$$y(x) = \int_x^{\infty} \frac{(x-t)^{n-1}}{(n-1)!} Q(t) y(t)\, dt.$$

The function $y(x)$ is bounded on $\langle c, \infty)$. Therefore from the last equation we have

$$\sup_{\langle c,\infty)} |y(x)| \leqslant \sup_{\langle c,\infty)} |y(x)| \int_c^{\infty} \frac{(t-c)^{n-1}}{(n-1)!} Q(t)\, dt.$$

But $\sup_{\langle c,\infty)} |y(x)| > 0$. This last inequality yields a contradiction to (20).

Let now

$$\int^{\infty} t^{n-1} Q(t)\, dt = \infty, \qquad \lim_{x\to\infty} y(x) = a > 0$$

and $y(x)$ have properties (B). Then there exists a number x_0 such that for $x \geqslant x_0$

$$y(x) > \frac{a}{2}. \tag{21}$$

By integrating equation (1) and using properties (B) we obtain

$$y'(x) = \int_x^{\infty} \frac{(x-t)^{n-2}}{(n-2)!} Q(t) y(t)\, dt,$$

and from this

$$-a+y(x) = \int_x^\infty \frac{(x-t)^{n-1}}{(n-1)!} Q(t)y(t)\,dt.$$

Now, according to (21) and the hypothesis, we obtain for $x \geqslant x_0$ the contradiction

$$-a+y(x) > \frac{1}{2}a\int_x^\infty \frac{(x-t)^{n-1}}{(n-1)!} Q(t)\,dt = \infty.$$

Definition. We shall say that $y(x)$ has properties (V) in (b, ∞) if for $x > b$

$$(-1)^i y^{(i)}(x) > 0 \text{ or } (-1)^{i+1} y^{(i)}(x) > 0, \qquad i = 0, 1, 2, \ldots, n-1,$$

$$\lim_{x\to\infty} y^{(i)}(x) = 0, \qquad i = 1, 2, \ldots, n-1,$$

$$\lim_{x\to\infty} y(x) \text{ exists and is not zero.}$$

THEOREM 3. *Let*

$$\int_x^\infty x^{n-1} Q(x)\,dx < \infty. \tag{22}$$

Then equation (1) *has only one solution* (*except for the linear dependence*) *having properties* (V) *in* (a, ∞).

Proof. The proof is by contradiction. Suppose $y_1(x) > 0$ and $y_2(x) > 0$ are two linearly independent solutions of (1) having properties (V). Then it follows from (22) and Theorem 2 that $\lim y_1(x) = a_1 > 0$ and $\lim y_2(x) = a_2 > 0$. But then the solution $y(x) = a_2 y_1(x) - a_1 y_2(x)$ has properties (B) and $\lim_{x\to\infty} y(x) = 0$ which contradicts (22).

II. Now we shall be concerned with the equation

$$y^{(n)} + B(x, y, y', \ldots, y^{(n-1)})y = 0. \tag{E}$$

THEOREM 4. 1° *Let* $B(x, u_0, u_1, \ldots, u_{n-1})$ *be a continuous and non-negative function on the region*

$$\Omega: \quad a < x < \infty, \qquad -\infty < u_i < \infty, \qquad i = 0, 1, \ldots, n-1,$$

such that for every point $(c_0, c_1, \ldots, c_{n-1}) \neq (0, 0, \ldots, 0)$ *the function* $B(x, c_0, c_1, \ldots, c_{n-1})$ *does not equal zero in any sub-interval of the interval* (a, ∞).

2° *Suppose there exists a continuous function* $F(x)$ *such that*

$$B(x, u_0, u_1, \ldots, u_{n-1}) \leqslant F(x)$$

for every point $(x, u_0, u_1, \ldots, u_{n-1}) \in \Omega$.

3° *Let*

$$\int^{\infty} x^{n-1} F(x)\, dx < \infty.$$

Then through every point (x_0, y_0), $y_0 \neq 0$, $a < x_0 < \infty$, *there passes at least one solution* $y(x)$ *of* (E) *having properties* (V) *in* (a, ∞).

Proof. We can suppose without loss of generality that $y_0 > 0$. Let D be a set of all functions $f(x)$ having continuous and bounded derivatives in the interval $\langle x_0, \infty)$ up to the order $n-1$ inclusive. Let

$$\|f(x)\|_D = \max_{0 \leqslant i \leqslant n-1} \{\sup_{\langle x_0, \infty)} |f^{(i)}(x)|\}$$

be a norm of $f(x)$. Then D is a Banach space.

Let, furthermore, C be a set of all functions $g(x)$ having continuous derivatives of the order $n-1$ in $\langle x_0, x_1\rangle$, and let

$$\|g(x)\|_C = \max_{0 \leqslant i \leqslant n-1} \{\max_{\langle x_0, x_1\rangle} |g^{(i)}(x)|\}$$

be a norm of $g(x)$.

Now, if $g(x) \in C$, let $\bar{g}(x) = \big(\eta_0(x), \eta_1(x), \ldots, \eta_{n-1}(x)\big)$ be a vector defined as follows:

$$\eta_i(x) = \begin{cases} g^{(i)}(x) & \text{for} \quad x \in \langle x_0, x_1\rangle, \\ g^{(i)}(x_1) & \text{for} \quad x > x_1, \end{cases} \qquad i = 0, 1, \ldots, n-1.$$

Let $\bar{C}$ denote the set of all such vectors $\bar{g}(x)$ belonging to $g(x) \in C$. Let $\|\bar{g}(x)\|_{\bar{C}} = \|g(x)\|_C$. C and $\bar{C}$ are Banach spaces. We define an operator $\bar{T}$ on $\bar{C}$ as follows: if $\bar{g}(x) \in \bar{C}$, then $\bar{T}\bar{g}(x) = u(x)$ is a solution of the equation

$$u^{(n)} + B\big(x, \eta_0(x), \eta_1(x), \ldots, \eta_{n-1}(x)\big) u = 0 \tag{23}$$

having properties (V) and passing through the point (x_0, y_0). Write $B\big(x, \eta_0(x), \eta_1(x), \ldots, \eta_{n-1}(x)\big) = B\big(x, \bar{g}(x)\big)$. Since $B\big(x, \bar{g}(x)\big)$ is a continuous function in $\langle x_0, \infty)$ and

$$\int_a^{\infty} x^{n-1} B\big(x, \bar{g}(x)\big)\, dx \leqslant \int_a^{\infty} x^{n-1} F(x)\, dx < \infty, \qquad a \geqslant x_0,$$

such solution $u(x)$ of (23) exists and is unique. It is clear that $\bar{T}\bar{g}(x) = u(x) \in D$.

Integrating (23) and using properties (V) of $u(x)$ we obtain

$$(24)\quad u^{(i)}(x) = \int_x^\infty \frac{(x-t)^{n-i-1}}{(n-i-1)!} B\big(t, \bar{g}(t)\big) u(t)\,dt, \qquad i = 1, 2, \ldots, n-1,$$

(25)

$$u(x) = y_0 - \int_{x_0}^\infty \frac{(x_0-t)^{n-1}}{(n-1)!} B\big(t, \bar{g}(t)\big) u(t)\,dt + \int_x^\infty \frac{(x-t)^{n-1}}{(n-1)!} B\big(t, \bar{g}(t)\big) u(t)\,dt.$$

From the positivity of y_0 and properties (V) of $u(x)$ it follows that $u(x)$ is decreasing in $\langle x_0, \infty)$ and $0 < u(x) \leqslant y_0$. Furthermore, in view of the assumptions 2° and 3°, (24) yields

$$(26)\quad |u^{(i)}(x)| \leqslant y_0 \int_{x_0}^\infty \frac{(t-x_0)^{n-i-1}}{(n-i-1)!} F(t)\,dt \leqslant y_0 \int_{x_0}^\infty (t-x_0+1)^{n-1} F(t)\,dt = K,$$
$$x \in \langle x_0, \infty), \qquad i = 1, 2, \ldots, n-1.$$

We also obtain

$$(27)\qquad \|\bar{T}\bar{g}(x)\|_D \leqslant \max\{y_0, K\} = S$$

for every $\bar{g}(x) \in \bar{C}$.

We shall prove the continuity of the operator $\bar{T}$. Let $\bar{g}_k(x) \in \bar{C}$, $\bar{g}(x) \in \bar{C}$, $\|\bar{g}_k(x) - \bar{g}(x)\|_{\bar{C}} \to 0$ as $k \to \infty$. Then we have to prove that $\|\bar{T}\bar{g}_k(x) - \bar{T}\bar{g}(x)\|_D \to 0$ as $k \to \infty$.

First we are going to prove that the sequence $\{\bar{T}\bar{g}_k(x)\}_{k=1}^\infty \equiv \{u_k(x)\}_{k=1}^\infty$ converges to $\bar{T}\bar{g}(x) = u(x)$ as $k \to \infty$. Since $0 < u_k(x) \leqslant y_0$ and $|u_k'(x)| \leqslant K$ for $x \geqslant x_0$ (see (26)), the functions $u_k(x)$ are uniformly bounded and equicontinuous on $\langle x_0, \infty)$. Also, the set of all functions $u_k(x)$, $k = 1, 2, \ldots$, is compact in any closed interval $\langle x_0, x_s\rangle$. It follows from this that it is possible to choose a subsequence $\{u_{kk}(x)\}_{k=1}^\infty$ of the sequence $\{u_k x\}_{k=1}^\infty$ such that there exists a positive function $v(x) \in D$ with $\lim\limits_{k\to\infty} u_{kk}(x) = v(x)$ for every $x \in \langle x_0, \infty)$ and $0 < v(x) \leqslant y_0$. One can prove this statement by constructing the subsequences $\{u_{sk}(x)\}_{k=1}^\infty$, $s = 1, 2, \ldots$, converging uniformly in $\langle x_0, x_s\rangle$ such that $\{u_{sk}(x)\}_{k=1}^\infty$ is a subsequence of $\{u_{s-1,k}(x)\}_{k=1}^\infty$. Then the diagonal sequence $\{u_{kk}(x)\}_{k=1}^\infty$ has the desired properties.

Let $u_{kk}(x) = \bar{T}\bar{g}_{kk}(x)$, $k = 1, 2, \ldots$ Clearly, $\|\bar{g}_{kk}(x) - \bar{g}(x)\|_{\bar{C}} \to 0$ as $k \to \infty$. Furthermore, the following formulae hold for $u_{kk}(x)$:

$$(28)\quad u_{kk}^{(i)}(x) = \int_x^\infty \frac{(x-t)^{n-i-1}}{(n-i-1)!} B\big(t, \bar{g}_{kk}(t)\big) u_{kk}(t)\,dt, \qquad i = 1, 2, \ldots, n-1,$$

$$(29)\qquad u(x) = y_0 - \int_{x_0}^{\infty} \frac{(x_0-t)^{n-1}}{(n-1)!} B\big(t, \bar{g}_{kk}(t)\big) u_{kk}(t)\,dt + \int_{x}^{\infty} \frac{(x-t)^{n-1}}{(n-1)!} B\big(t, \bar{g}_{kk}(t)\big) u_{kk}(t)\,dt.$$

Since $B\big(t, \bar{g}_{kk}(t)\big) u_{kk}(t)$ converges to $B\big(t, \bar{g}(t)\big) v(t)$ for every $x \in \langle x_0, \infty)$ and the functions under the sign of integration in (28) and (29) have integrable majorants $(t-x_0)^{n-i-1} F(t) y_0$, we obtain from (28) and (29) the equations

$$v(x) = y_0 - \int_{x_0}^{\infty} \frac{(x_0-t)^{n-1}}{(n-1)!} B\big(t, \bar{g}(t)\big) v(t) + \int_{x}^{\infty} \frac{(x-t)^{n-1}}{(n-1)!} B\big(t, \bar{g}(t)\big) v(t)\,dt,$$

$$\lim_{k\to\infty} u_{kk}^{(i)}(x) = \int_{x}^{\infty} \frac{(x-t)^{n-i-1}}{(n-i-1)!} B\big(t, \bar{g}(t)\big) v(t)\,dt = v^{(i)}(x), \qquad i = 1, 2, \ldots, n-1.$$

which means that

$$\lim_{k\to\infty} u_{kk}^{(i)}(x) = v^{(i)}(x) = \big(\bar{T}\bar{g}(x)\big)^{(i)} = u^{(i)}(x), \qquad i = 1, 2, \ldots, n-1.$$

The above reasoning allows to formulate the statement: from every subsequence of the sequence $\{u_k(x)\}_{k=1}^{\infty}$ it is possible to choose a subsequence converging to $u(x)$ for every $x \in \langle x_0, \infty)$. But this means that the sequence $u_k(x)$ converges to $u(x)$ for every $x \in \langle x_0, \infty)$.

Now, it is easy to prove that $\|u_k(x) - u(x)\|_D \to 0$ as $k \to \infty$. From the formulae similar to (28) and (29) holding for $u_k^{(i)}(x)$ and $u^{(i)}(x)$, $i = 0, 1, 2, \ldots, n-1$, we obtain

$$|u_k^{(i)}(x) - u^{(i)}(x)| \leqslant \int_{x_0}^{\infty} \frac{(t-x_0)^{n-i-1}}{(n-i-1)!} \big|B\big(t, \bar{g}_k(t)\big) u_k(t) - B\big(t, \bar{g}(t)\big) u(t)\big|\,dt$$

$$\leqslant \int_{x_0}^{\infty} (t-x_0+1)^{n-1} \big|B\big(t, \bar{g}_k(t)\big) u_k(t) - B\big(t, \bar{g}(t)\big) u(t)\big|\,dt, \qquad i = 1, 2, \ldots, n-1,$$

$$|u_k(x) - u(x)| \leqslant 2 \int_{x_0}^{\infty} (t-x_0+1)^{n-1} \big|B\big(t, \bar{g}_k(t)\big) u_k(t) - B\big(t, \bar{g}(t)\big) u(t)\big|\,dt$$

for $x \in \langle x_0, \infty)$, and

$$\|u_k(x) - u(x)\|_D \leqslant \max_{0 \leqslant i \leqslant n-1} \{\sup_{\langle x_0, \infty)} |u_k^{(i)}(x) - u^{(i)}(x)|\}$$

$$\leqslant 2 \int_{x_0}^{\infty} (t-x_0+1)^{n-1} \big|B\big(t, \bar{g}_k(t)\big) u_k(t) - B\big(t, \bar{g}(t)\big) u(t)\big|\,dt.$$

The same reasoning as above yields

$$\lim_{k\to\infty} \|u_k(x) - u(x)\|_D = \lim_{k\to\infty} \|\overline{T}\bar{g}_k(x) - \overline{T}\bar{g}(x)\|_D = 0.$$

Thus the continuity of $\overline{T}$ is proved.

Define an operator T_1 on C as follows: If $g(x) \in C$, then $T_1 g(x) = \overline{T}\bar{g}(x)$ for $x \in \langle x_0, x_1\rangle$. From this definition it is evident that $\|T_1 g(x)\|_C \leqslant \|\overline{T}\bar{g}(x)\|_D$. Furthermore, the continuity of T_1 follows from the continuity of $\overline{T}$, for

$$\|T_1 g_k(x) - T_1 g(x)\|_C \leqslant \|\overline{T}\bar{g}_k(x) - \overline{T}\bar{g}(x)\|_D$$

and

$$\|g_k(x) - g(x)\|_C = \|\bar{g}_k(x) - \bar{g}(x)\|_{\bar{C}}.$$

Let S be the number from (27) and let

(30) $$M = \{g(x) \in C \mid \|g(x)\|_C \leqslant S\}, \quad \overline{M} = \{\bar{g}(x) \in \bar{C} \mid g(x) \in M\}.$$

According to (27) we have $\|T_1 g(x)\|_C \leqslant \|\overline{T}\bar{g}(x)\|_D \leqslant S$, so $T_1 M \subset M$. Let $\bar{g}(x) \in \overline{M}$ be an arbitrary element. Then

$$\max_{\langle x_0, x_1\rangle} B\big(x, \bar{g}(x)\big) \leqslant \max_{\langle x_0, x_1\rangle} F(x) = N.$$

Thus from (23) we obtain

(31) $$|u^{(n)}| \leqslant N y_0.$$

This fact and the fact that $\|u(x)\|_D \leqslant S$ imply the compactness of $T_1 M$. According to the Schauder theorem, T_1 has (at least one) fixed point in M, i.e. there exists an element $g^*(x) \in M$ such that $T_1 g^*(x) = g^*(x)$ and $g^*(x) = \overline{T}\bar{g}^*(x)$ on $\langle x_0, x_1\rangle$. $\overline{T}\bar{g}^*(x)$ has properties (V) in $\langle x_0, \infty)$ and $\big(\overline{T}\bar{g}(x)\big)(x_0) = y_0 = g^*(x_0)$.

Let now $x_0 < x_1 < x_2 < \ldots < x_k \ldots$, $\lim x_k = \infty$ as $k \to \infty$. For every k we construct in $\langle x_0, x_k\rangle$ the spaces C_k, $\bar{C}_k$, the sets M_k, $\overline{M}_k$ and the operators T_k, $\overline{T}_k$. Let $g_k(x) \in M_k$ be a fixed point of T_k and let $\overline{T}_k \bar{g}_k^*(x) = z_k(x)$. We know that

(32) $$g_k^*(x) = T_k g_k^*(x), \quad g_k^*(x) = \overline{T}_k \bar{g}_k^*(x) \quad \text{for} \quad x \in \langle x_0, x_k\rangle.$$

Let $\bar{g}_k^*(x) = \big([\eta_0^*(x)]_k, [\eta_1^*(x)]_k, \ldots, [\eta_{n-1}^*(x)]_k\big)$. Construct the following sequences:

(33) $$\{[\eta_i^*(x)]_k\}_{k=1}^{\infty}, \quad \{z_k^{(i)}(x)\}, \quad i = 0, 1, \ldots, n-1.$$

From (32) we have

(34) $$[\eta_i^*(x)]_k = z_k^{(i)}(x), \quad i = 0, 1, \ldots, n-1,$$

for $x \in \langle x_0, x_k\rangle$. All functions of sequences (33) are uniformly bounded by S. From that and from the inequality like (31) we conclude that sequences (33) are compact in any finite closed interval $\langle x_0, x_s\rangle$. It is also possible to choose a subsequence $\{[\eta_i^*(x)]_{1k}\}_{k=1}^{\infty}$ and $\{z_{1k}^{(i)}(x)\}_{k=1}^{\infty}$ uniformly convergent in $\langle x_0, x_1\rangle$ to continuous functions $v_1^{(i)}(x)$, $i = 0, 1, \ldots, n-1$. From $\{[\eta_i^*(x)]_{1k}\}_{k=1}^{\infty}$ and $\{z_{1k}^{(i)}(x)\}_{k=1}^{\infty}$ we can choose subsequences $\{[\eta_i^*(x)]_{2k}\}_{k=1}^{\infty}$ and $\{z_{2k}^{(i)}(x)\}_{k=1}^{\infty}$ convergent uniformly in $\langle x_0, x_2\rangle$ to continuous functions $v_2^{(i)}(x)$. It is clear that $v_1^{(i)}(x) = v_2^{(i)}(x)$ for $x \in \langle x_0, x_1\rangle$, $i = 0, 1, \ldots, n-1$. Repeating this procedure we obtain the sequences

$$\{[\eta_i^*(x)]_{jk}\}_{k=1}^{\infty},\ \{z_{jk}^{(i)}(x)\},\ i = 0, 1, \ldots, n-1,\ j = 1, 2, \ldots,$$

such that $\{[\eta_i^*(x)]_{jk}\}_{k=1}^{\infty}$ and $\{z_{jk}^{(i)}(x)\}_{k=1}^{\infty}$ converge uniformly in $\langle x_0, x_j\rangle$ to continuous functions $v_j^{(i)}(x)$, $i = 0, 1, \ldots, n-1$. We have $v_j^{(i)}(x) = v_{j+1}^{(i)}(x) = \ldots$ in $\langle x_0, x_j\rangle$, $i = 0, 1, \ldots, n-1$. Now, if we construct the diagonal sequences $\{[\eta_i(x)]_{kk}\}_{k=1}^{\infty}$, $\{z_{kk}^{(i)}(x)\}_{k=1}^{\infty}$, these sequences converge to $v^{(i)}(x) = \lim\limits_{j\to\infty} v_j^{(i)}(x)$ for every $x \in \langle x_0, \infty)$, i.e.

$$\lim_{k\to\infty} z_{kk}^{(i)}(x) = v^{(i)}(x) = \lim_{k\to\infty} [\eta_i^*(x)]_{kk}, \qquad i = 0, 1, \ldots, n-1. \tag{35}$$

But for $z_{kk}(x)$ we have the formulae

$$z_{kk}(x) = y_0 - \int_{x_0}^{\infty} \frac{(x_0-t)^{n-1}}{(n-1)!} B\big(t, [\eta_0^*(t)]_{kk}, [\eta_1^*(t)]_{kk}, \ldots, [\eta_{n-1}^*(t)]_{kk}\big) z_{kk}(t)\,dt +$$

$$+ \int_{x}^{\infty} \frac{(x-t)^{n-1}}{(n-1)!} B\big(t, [\eta_0^*(t)]_{kk}, [\eta_1^*(t)]_{kk}, \ldots, [\eta_{n-1}^*(t)]_{kk}\big) z_{kk}(t)\,dt,$$

$$z_{kk}^{(i)}(x) = \int_{x}^{\infty} \frac{(x-t)^{n-i-1}}{(n-i-1)!} B\big(t, [\eta_0^*(t)]_{kk}, [\eta_1^*(t)]_{kk}, \ldots, [\eta_{n-1}^*(t)]_{kk}\big) z_{kk}(t)\,dt,$$

$$i = 1, 2, \ldots, n-1.$$

Using (35) and the Lebesgue's theorem we obtain

$$v(x) = y_0 - \int_{x_0}^{\infty} \frac{(x_0-t)^{n-1}}{(n-1)!} B\big(t, v(t), v'(t), \ldots, v^{(n-1)}(t)\big) v(t)\,dt +$$

$$+ \int_{x}^{\infty} \frac{(x-t)^{n-1}}{(n-1)!} B\big(t, v(t), v'(t), \ldots, v^{(n-1)}(t)\big) v(t)\,dt,$$

$$v^{(i)}(x) = \int_{x}^{\infty} \frac{(x-t)^{n-i-1}}{(n-i-1)!} B\big(t, v(t), v'(t), \ldots, v^{(n-1)}(t)\big) v(t)\,dt,$$

$$i = 1, 2, \ldots, n-1.$$

It is easy to see that $v(x)$ is a solution of (E) having the properties (V) in $\langle x_0, \infty)$ and passing through the point (x_0, y_0).

It remains to prove that $v(x)$ can be extended to (a, ∞). It is evident that $v(x)$ can be extended. Let $y(x)$ be the extension of $v(x)$ to the largest interval (b, ∞), $a < b < x_0$. It follows from properties (V) that

$$(-1)^i y^{(i)}(x_0) > 0, \quad i = 0, 1, \ldots, n-1. \tag{36}$$

Make the change of variable by substituting $t = x_0 - x$. Then, if $x \in (a, x_0\rangle$, $t \in \langle 0, x_0 - a)$. Furthermore, we obtain the relations

$$y(x) = y(x_0 - t) = p(t), \quad y^{(i)}(x) = (-1)^i p^{[i]}(x), \quad i = 0, 1, \ldots, n, \tag{37}$$

where $p^{[i]}(t)$ denotes $d^i p(t)/dt^i$. From (E) we obtain for $p(t)$ the equation

$$p^{[n]}(t) - B\big(x_0 - t, p(t), -p^{[1]}(t), \ldots, p^{[n-1]}(t)\big) p(t) = 0. \tag{38}$$

The initial conditions for $p(t)$ are

$$p^{[i]}(0) = (-1)^i y^{(i)}(x_0) > 0. \tag{39}$$

Let $B\big(x_0 - t, p(t), -p^{[1]}(t), \ldots, p^{[n-1]}(t)\big) = B_*\big(t, p(t), p^{[1]}(t), \ldots p^{[n-1]}(t)\big)$. $B_*(t, p_0, p_1, \ldots, p_{n-1})$ is a continuous and non-negative function in Ω_1: $-\infty < t < x_0 - a$, $-\infty < p_i < \infty$, $i = 0, 1, \ldots, n-1$, and $0 \leqslant B(t, p_0, p_1, \ldots, p_{n-1}) \leqslant F(x_0 - t) = F_*(t)$ for every point $(t, p_0, p_1, \ldots \ldots, p_{n-1}) \in \Omega_1$. $F_*(t)$ is continuous in $(-\infty, x_0 - a)$. We seek a solution of the differential equation

$$p^{[n]} = B_*(t, p, p^{[1]}, \ldots, p^{[n-1]}) p \tag{40}$$

determined by the initial conditions

$$p^{[i]}(0) > 0, \quad i = 0, 1, \ldots, n-1. \tag{41}$$

It is easy to see that there exists a solution $p(t)$ in the interval $(-\infty, x_0 - b)$ and in this interval the derivatives $p^{[i]}(t)$, $i = 0, 1, \ldots, n-1$, are positive increasing functions. This follows from properties (V) of $y(x)$ in $\langle x_0, \infty)$ and from equation (40) and initial conditions (41). Furthermore, it is evident that $p^{[i]}(t)$, $i = 0, 1, \ldots, n-1$, are positive increasing functions in the whole interval of the existence of $p(t)$. We shall prove that this interval is the interval $(-\infty, x_0 - a)$. Suppose that $a < b$ and therefore $x_0 - b < x_0 - a$ and suppose that $p(t)$ cannot be extended to the interval $\langle 0, x_0 - b\rangle$. This means that $\lim p(t) = \infty$ as $t \to (x_0 - b)^-$. But by integrating (40) we obtain

$$p(t) = \sum_{i=0}^{n-1} \frac{p^{[i]}(0)}{i!} t^i + \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} B_*\big(\tau, p(\tau), \ldots, p^{[n-1]}(\tau)\big) p(\tau)\, d\tau,$$

for $t \in \langle 0, x_0-b)$. From this we get

$$(42) \qquad p(t) \leqslant C + \int_0^t (x_0-b-\tau)^{n-1} F_*(\tau) p(\tau)\, d\tau,$$

where

$$C = \sum_{i=0}^{n-1} \frac{p^{[i]}(0)}{i!} (x_0-b)^i.$$

Now, the Gronwall-Bellman lemma yields

$$p(t) \leqslant C \exp \int_0^t (x_0-b-\tau)^{n-1} F_*(\tau)\, d\tau.$$

Since $F_*(t)$ is continuous on $\langle 0, x_0-b\rangle$, we see that $\lim p(t) < \infty$ as $t \to (x_0-b)^-$. This is a contradiction which proves that $p(t)$ can be extended to $\langle 0, x_0-b\rangle$. But this implies the existence of $p(t)$ in $(-\infty, x_0-a)$ and therefore the existence of $y(x)$ in (a, ∞). Now, it is clear that $y(x)$ is a solution of (E) having properties (V) in (a, ∞) and passing through the point (x_0, y_0).

THEOREM 5. *Let the hypotheses of Theorem 4 be satisfied. Then to every real number $m_0 \neq 0$ there exists a solution $y(x)$ of (E) having properties (V) in (a, ∞) and such that*

$$\lim_{x\to\infty} y(x) = m_0 \quad \text{as} \quad x \to \infty.$$

Proof. Without loss of generality we can suppose that $m_0 > 0$. Let (x_0, y_0), $y_0 > 0$, $x_0 \in (a, \infty)$, be an arbitrary point. According to Theorem 4 through this point there passes a solution $u(x)$ of (E) having properties (V) in (a, ∞). We have

$$u(x) = y_0 - \int_{x_0}^{\infty} \frac{(x_0-t)^{n-1}}{(n-1)!} B\big(t, u(t), u'(t), \ldots, u^{(n-1)}(t)\big) u(t)\, dt +$$
$$+ \int_x^{\infty} \frac{(x-t)^{n-1}}{(n-1)!} B\big(t, u(t), u'(t), \ldots, u^{(n-1)}(t)\big) u(t)\, dt$$

and

$$\lim_{x\to\infty} u(x) = d = y_0 - \int_{x_0}^{\infty} \frac{(x_0-t)^{n-1}}{(n-1)!} B\big(t, u(t), u'(t), \ldots, u^{(n-1)}(t)\big) u(t)\, dt.$$

It is clear that $d < y_0$. On the other hand, we obtain the estimate

$$y_0 - d \leqslant y_0 \int_{x_0}^{\infty} \frac{(x_0-t)^{n-1}}{(n-1)!} F(t)\, dt = y_0 L,$$

where

$$L = \int_{x_0}^{\infty} \frac{(x_0-t)^{n-1}}{(n-1)!} F(t)\,dt \tag{43}$$

depends only on x_0. For d we obtain the estimate

$$y_0(1-L) \leqslant d < y_0. \tag{44}$$

Let U be the set of all those numbers d from $(0, m_0\rangle$ for which there exists a solution $u(x)$ of (E) having properties (V) in (a, ∞) such that $\lim u(x) = d$ as $x \to \infty$. We shall prove that $\sup U = m_0$. For suppose we have $\sup U < m_0$. Then put $y_0 = m_0$ and choose x_0 such that $\sup U > m_0(1-L)$. It follows from (43) that this is possible. Through the point (x_0, m_0) there passes a solution $u(x)$ of (E) having properties (V) in (a, ∞) such that for $d = \lim u(x)$ we have (according to (44))

$$\sup U < m_0(1-L) \leqslant d < m_0.$$

This is a contradiction which proves that $\sup U = m_0$.

Now, there are two possible cases:

1) $m_0 \in U$. But this means that there exists a solution $u(x)$ of (E) having properties (V) in (a, ∞) and such that $\lim u(x) = m_0$ as $x \to \infty$. In this case the Theorem is proved.

2) $m_0 \notin U$. Then there exists a sequence $\{d_k\}_{k=1}^{\infty}$, $0 < \varepsilon < d_k < m_0$, $d_k \in U$, converging to m_0 as $k \to \infty$. Let $y_k(x)$ be a solution of (E) having properties (V) in (a, ∞) such that $\lim y_k(x) = d_k$ as $x \to \infty$. Choose x_0 such that $L < 1$ and consider the sequence $\{y_k(x_0)\}_{k=1}^{\infty}$. It is clear that $y_k(x_0) > \varepsilon$. We prove that $y_k(x_0) \leqslant m_0/(1-L)$. Let the inequality $y_{k_1}(x_0) > m_0/(1-L)$ be satisfied for k_1. Then the solution of (E) having properties (V) in (a, ∞) and passing through the point $\big(x_0, y_{k_1}(x_0)\big)$ has a limit d_{k_1} as $x \to \infty$ for which (according to (44)) we have $y_{k_1}(x_0)(1-L) \leqslant d_{k_1}$. From this inequality we obtain a contradiction

$$m_0 = m_0(1-L)/(1-L) < y_{k_1}(x_0)(1-L) \leqslant d_{k_1}.$$

This contradiction proves that the sequence $\{y_k(x_0)\}_{k=1}^{\infty}$ is also bounded from above. Therefore we can choose a subsequence $\{y_{1k}(x_0)\}_{k=1}^{\infty}$ which is convergent. Suppose we have $\lim\limits_{k\to\infty} y_{1k}(x_0) = y_0$. It is clear that $\varepsilon \leqslant y_0 \leqslant m_0/(1-L)$. Now, consider the sequence $\{y_{1k}(x)\}_{k=1}^{\infty}$. Suppose that $\lim y_{1k}(x) = d_{lk}$ as $x \to \infty$. The sequence $\{d_{1k}\}_{k=1}^{\infty}$ is a subsequence of $\{d_k\}_{k=1}^{\infty}$ and therefore $\lim d_{1k} = m_0$ as $k \to \infty$.

Furthermore, for $y_{1k}(x)$ there hold the relations

$$(45)\qquad y_{1k}(x) = y_{1k}(x_0) - \int_{x_0}^{\infty} \frac{(x_0-t)^{n-1}}{(n-1)!} B\big(t, y_{1k}(t), \ldots, y_{1k}^{(n-1)}(t)\big) y_{1k}(t) + $$

$$+ \int_{x_0}^{\infty} \frac{(x-t)^{n-1}}{(n-1)!} B\big(t, y_{1k}(t), \ldots, y_{1k}^{(n-1)}(t)\big) y_{1k}(t)\, dt,$$

$$(46)\qquad y_{1k}^{(i)}(x) = \int_{x}^{\infty} \frac{(x-t)^{n-i-1}}{(n-i-1)!} B\big(t, y_{1k}(t), \ldots, y_{1k}^{(n-1)}(t)\big) y_{1k}(t)\, dt,$$

$$i = 1, 2, \ldots, n-1.$$

It is easy to see that $0 < y_{1k}(x) \leqslant m_0/(1-L)$ and

$$|y^{(i)}(x)| \leqslant \frac{m_0}{1-L} \int_{x_0}^{\infty} (t-x_0+1)^{n-1} F(t)\, dt = S_1, \qquad i = 1, 2, \ldots, n-1.$$

From this we get $\|y_{1k}(x)\|_D \leqslant \max\{m_0/(1-L), S_1\}$, which means that the functions $y_{1k}^{(i)}(x)$, $i = 0, 1, \ldots, n-1$, $k = 1, 2, \ldots$, are uniformly bounded in $\langle x_0, \infty)$. It follows from this that the functions $y_{1k}^{(i)}(x)$, $i = 0, 1, \ldots, n-2$, $k = 1, 2, \ldots$, are equicontinuous in $\langle x_0, \infty)$. But from the relation

$$|y_{1k}^{(n-1)}(x)| \leqslant \frac{m_0}{1-L} \int_{x_0}^{\infty} (t-x_0+1)^{n-1} F(t)\, dt < \infty$$

we get that the functions $y_{1k}^{(n-1)}(x)$ are also equicontinuous in $\langle x_0, \infty)$. Now, using the same reasoning as in the proof of Theorem 4 one can choose a subsequence $\{y_{kk}(x)\}_{k=1}^{\infty}$ of the sequence $\{y_{1k}(x)\}_{k=1}^{\infty}$ such that

$$(47)\qquad \lim_{k\to\infty} y_{kk}(x) = y(x), \qquad \lim_{k\to\infty} y_{kk}^{(i)}(x) = y^{(i)}(x), \qquad i = 1, 2, \ldots, n-1,$$

for $x \in \langle x_0, \infty)$. Furthermore,

$$(48)\qquad \lim_{k\to\infty} y_{kk}(x_0) = y_0, \qquad \lim_{k\to\infty} \lim_{k\to\infty} y_{kk}(x) = \lim_{k\to\infty} d_{kk} = m_0.$$

For $y_{kk}(x)$ we have formulae like (45) and (46). Then, using (47) and (48) and the Lebesgue theorem, we get

$$(49)\qquad y(x) = y_0 - \int_{x_0}^{\infty} \frac{(x_0-t)^{n-1}}{(n-1)!} B\big(t, y(t), y'(t), \ldots, y^{(n-1)}(t)\big) y(t)\, dt + $$

$$+ \int_{x}^{\infty} \frac{(x-t)^{n-1}}{(n-1)!} B\big(t, y(t), y'(t), \ldots, y^{(n-1)}(t)\big) y(t)\, dt,$$

$$(50)\qquad y^{(i)}(x) = \int\limits_{x}^{\infty} \frac{(x-t)^{n-i-1}}{(n-i-1)!} B\big(t, y(t), y'(t), \ldots, y^{(n-1)}(t)\big) y(t)\, dt,$$

$$i = 1, 2, \ldots, n-1.$$

This means that $y(x)$ is a solution of (E) having properties (V) in (x_0, ∞) and passing through the point (x_0, y_0). Furthermore, the same reasoning as that used in the proof of the continuity of $\overline{T}$ in the proof of Theorem 4 yields $\|y_{kk}(x) - y(x)\|_D \to 0$ as $k \to \infty$. Thus y_{kk} converges uniformly to $y(x)$ in $\langle x_0, \infty)$. Therefore

$$m_0 = \lim_{k\to\infty} \lim_{k\to\infty} y_{kk}(x) \doteq \lim_{k\to\infty} \lim_{k\to\infty} y_{kk}(x) = \lim_{k\to\infty} y(x).$$

According to Theorem 4, $y(x)$ can be extended over (a, ∞) with the conservation of properties (V). This concludes the proof of Theorem 5.

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