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ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF A BOUNDARY VALUE PROBLEM FOR AN ORDINARY SECOND-ORDER DIFFERENTIAL EQUATION

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It is well known [2] that for the linear differential equations the uniqueness of solutions of a boundary value problem implies their existence. For the non-linear differential equations this interdependence of uniqueness and existence is much more complicated and involves, in general, besides the non-linear equation under consideration, an appropriate family of linear equations [1], [3], [4].

However, as we shall show in the present paper, in the special case of the non-linear second-order differential equation it is possible to infer the existence of solutions of a boundary value problem immediately from the uniqueness of solutions of this problem for the equation itself, without recurring to a comparative family of linear equations.

In Section 1 we formulate our main theorem. Section 2 is devoted to a discussion on the assumptions of this theorem and Section 3 contains its proof. In the last section we indicate some generalizations.

1. Consider a differential equation

$$x'' = f(t, x, x') \tag{1}$$

and assume that the real function $f(t, x, u)$ defined in the strip

$$D = (a, b) \times R^2$$

(R denotes the real line) satisfies the following condition:

(C) For every point $(t_0, x_0, u_0) \in D$ there exists one and only one solution $x(t) = x(t; t_0, x_0, u_0)$ of equation (1), defined on (a, b) and such that $x(t_0) = x_0$, $x'(t_0) = u_0$.

Moreover, consider a boundary value condition

$$x(t_1) = r_1, \quad x(t_2) = r_2 \qquad (a < t_1 < t_2 < b). \tag{2}$$

THEOREM 1. *If the function $f(t, x, u)$ is continuous in the strip D, satisfies condition* (C) *and for every pair (t_1, r_1), (t_2, r_2) of points of the set $(a, b)\times R$ $(t_1 < t_2)$ there exists at most one solution of problem* (1), (2), *then for each such pair there exists one and only one solution of this problem.*

2. Before proceeding to the proof of Theorem 1 let us observe that it does not hold for the closed interval $[a, b]$. In order to prove this, denote by $\varphi(p, q)$ the solution of the equation

$$\varphi + \frac{p}{2}\operatorname{arctg}\varphi = q \qquad (p > -2). \tag{3}$$

It is easily seen that the family of all solutions of the differential equation

$$x'' = -x + \tfrac{1}{2}\operatorname{arctg}\varphi(\sin t, x\sin t + x'\cos t) \tag{4}$$

is given by the formula

$$x(t) = A\cos t + B\sin t + \tfrac{1}{2}\operatorname{arctg}B,$$

where A and B are arbitrary constants. Thus, the boundary value condition (2) for equation (4) leads to the following system of equations:

$$A\cos t_i + B\sin t_i + \tfrac{1}{2}\operatorname{arctg}B = r_i \qquad (i = 1, 2).$$

After elimination of A we have

$$B\sin(t_2 - t_1) + \tfrac{1}{2}(\cos t_1 - \cos t_2)\operatorname{arctg}B = r_2\cos t_1 - r_1\cos t_2. \tag{5}$$

If $0 \leqslant t_1 < t_2 < \pi$ or $0 < t_1 < t_2 \leqslant \pi$, then obviously

$$\sin(t_2 - t_1) > 0, \qquad \cos t_1 - \cos t_2 > 0.$$

Hence it follows immediately that for every pair r_1, r_2 of real numbers the problem (4), (2) has a uniquely determined solution. However, when we set $t_1 = 0$ and $t_2 = \pi$, equation (5) reduces to

$$\operatorname{arctg}B = r_1 + r_2.$$

As before, this assures the uniqueness of solutions of problem (4), (2) but at the same time it proves that they exist only if

$$|r_1 + r_2| < \pi/2.$$

3. Passing now to the proof of Theorem 1, fix the points (t_1, r_1), (t_2, r_2) and, for an arbitrary $u \in R$, denote by $x(t, u)$ the solution of (1) satisfying $x(t_1) = r_1$ and $x'(t_1) = u$. From assumption (C) it follows that the mapping $T\colon R \to R$ defined by the formula $T(u) = x(t_2, u)$ is continuous. Similarly, from the uniqueness of solutions of problem (1), (2) it immediately follows that T is an injection. Hence its range $T(R)$ is an open and connected subset of R, i. e. an open (finite or infinite) interval.

Thus, in order to prove that $T(R) = R$, it remains only to show that $\sup T(R) = +\infty$ and $\inf T(R) = -\infty$. Suppose that $p_0 = \sup T(R) < +\infty$ and choose an increasing sequence $\{p_n\} \subset T(R)$ converging to p_0. By setting $u_n = T^{-1}(p_n)$ and $x_n(t) = x(t, u_n)$ we get

$$x_n(t_1) = r_1, \quad x_n(t_2) = p_n. \tag{6}$$

From the uniqueness of solutions of problem (1), (2) it follows that

$$x_n(t) > x_1(t) \quad (t_1 < t < b, \quad n = 2, 3, \ldots). \tag{7}$$

For infinitely many values of n we have either $x_n'(t_2) \leqslant 0$ or $x_n'(t_2) \geqslant 0$. We shall consider only the first of these cases, for the other presents no further difficulties. Passing to an appropriate subsequence, if necessary, we may assume without loss of generality that

$$x_n'(t_2) \leqslant 0 \quad (n = 1, 2, \ldots). \tag{8}$$

Let t_3 be a fixed point belonging to (t_2, b). From (7) it easily follows that

$$\frac{x_n(t_3) - x_n(t_2)}{t_3 - t_2} \geqslant \frac{x_1(t_3) - x_n(t_2)}{t_3 - t_2} \geqslant K = \min\left(0, \frac{x_1(t_3) - p_0}{t_3 - t_2}\right).$$

From this inequality and from (8) it follows that for every $n = 1, 2, \ldots$ the set

$$S_n = \{t\colon t_2 \leqslant t \leqslant t_3, \ K \leqslant x_n'(t) \leqslant 0\}$$

is non-empty. Setting $s_n = \min S_n$, we have $x_n'(t) \leqslant 0$ for $t_2 \leqslant t \leqslant s_n$ and therefore $x_n(s_n) \leqslant x_n(t_2) \leqslant p_0$. On the other hand, by (7) we have

$$L = \min_{[t_2, t_3]} x_1(t) \leqslant x_1(s_n) \leqslant x_n(s_n),$$

so that

$$L \leqslant x_n(s_n) \leqslant p_0, \quad K \leqslant x_n'(s_n) \leqslant 0, \quad t_2 \leqslant s_n \leqslant t_3.$$

Replacing, if necessary, the sequence $\{s_n\}$ by an appropriate subsequence, we may assume that there exist the limits

$$s_0 = \lim_{n\to\infty} s_n, \quad x_0 = \lim_{n\to\infty} x_n(s_n), \quad u_0 = \lim_{n\to\infty} x_n'(s_n).$$

From the continuous dependence of solutions of (1) on their initial values it follows that the sequence $\{x_n(t)\}$ converges in (a, b) to a solution

$x_0(t)$ of equation (1) such that $x_0(s_0) = x_0$ and $x_0'(s_0) = u_0$. Moreover, by (6) we have

$$x_0(t_1) = r_1, \quad x_0(t_2) = p_0.$$

This means that $p_0 = T\big(x_0'(t_1)\big)$, so that $p_0 \in T(R)$. But this is impossible, since $T(R)$ is open, and therefore $\sup T(R) = +\infty$.

The proof that $\inf T(R) = -\infty$ is quite similar and will be left to the reader.

4. Theorem 1 remains true if we replace the boundary value condition (2) by a more general condition

(9) $\quad \alpha x(t_1) + \beta x'(t_1) = r_1, \quad x(t_2) = r_2 \quad (a < t_1 \neq t_2 < b;\ \alpha^2 + \beta^2 > 0).$

We have then the following

THEOREM 2. *If the function $f(t, x, u)$ is continuous in the strip D, satisfies condition (C) and for every pair of points (t_1, r_1) and (t_2, r_2) of $(a, b) \times R$ $(t_1 \neq t_2)$ there exists at most one solution of problem (1), (9), then for every such pair of points there exists one and only one solution of this problem.*

The proof of this theorem is quite analogous to that of Theorem 1, the only difference lies in the definition of $x(t, u)$ which denotes now the solution of equation (1) satisfying the initial conditions

$$x(t_1) = u, \quad x'(t_1) = \frac{1}{\beta}(r_1 - \alpha u).$$

It is worth while to notice that Theorem 2 does not hold if we replace condition (9) by a slightly more general condition

$$\alpha_i x(t_i) + \beta_i x'(t_i) = r_i \quad (t_1 \neq t_2,\ \alpha_i^2 + \beta_i^2 > 0,\ i = 1, 2).$$

In order to prove this, consider the differential equation

(10) $$x'' = e^t \varphi(2e^{-t}, x' e^{-t}),$$

where φ denotes the function defined by (3), and the boundary value condition

(11) $$x'(t_1) - x(t_1) = r_1, \quad x'(t_2) - x(t_2) = r_2 \quad (t_1 \neq t_2).$$

Since the family of solutions of (10) is given by the formula

$$x(t) = A + Be^t + t \operatorname{arc\,tg} B,$$

in which A and B are arbitrary constants, conditions (11) lead to the system of equations

$$(1 - t_i) \operatorname{arc\,tg} B - A = r_i \quad (i = 1, 2).$$

The elimination of A yields

$$(t_1 - t_2)\operatorname{arc\,tg} B = r_2 - r_1.$$

Thus problem (10), (11) has at most one solution, but for

$$|r_2 - r_1| \geqslant \frac{\pi}{2} |t_2 - t_1|$$

the solution does not exist.

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