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ON TWO THEOREMS OF DYER

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In [1] and [2] E. Dyer proved several interesting results on continuous decompositions of continua into decomposable elements. In this paper we discuss some of those results again. We are doing this because of their importance for continua theory and because we have found either simpler proofs or some generalizations of them. The author believes that the generalizations of Dyer's results from [2] which we obtain in Section 3 completely characterize the quotient spaces of continuous decompositions of planar continua into decomposable continua (comp. Problem 1 in Section 4).

1. Terminology and auxiliary results. In this section we establish our notation, prove some simple results and recall some known results for purpose of future references.

All spaces under considerations are assumed to be metrizable and, in fact, we usually assume that the spaces are equipped with fixed metrics. A compact connected nonvoid space is called a *continuum*. A continuum is said to be:

(i) *completely regular* if each nondegenerate subcontinuum has nonvoid interior;

(ii) *regular* if the continuum has a base of open sets with finite boundaries;

(iii) *finitely suslinian* if each sequence of pairwise disjoint subcontinua forms a null sequence, i.e. the diameters of the subcontinua converge to zero;

(iv) *hereditarily locally connected* if each subcontinuum is locally connected;

(v) *suslinian* if each collection of nondegenerate pairwise disjoint subcontinua is countable;

(vi) *hereditarily decomposable* if each nondegenerate subcontinuum is decomposable.

PROPOSITION 1.1. *The following implications hold true*:

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(v)} \Rightarrow \text{(vi)}$$

and none of them can be reversed.

The first implication is proved in [4], Theorem 3, p. 284; the others are easily verifiable. For more details and appropriate counterexamples we refer the reader to [4] and [5].

Since completely regular continua play an important role in this note we shall give another, more geometrical, characterization of this class. To this end we need some more terminology.

If A is an arc, then $\dot{A}$ denotes its boundary, and $\mathring{A} = A \setminus \dot{A}$. Let A_1, A_2, A_3 be three disjoint arcs and let B be a three-point set obtained by selecting one point from each of the sets $\dot{A}_1$, $\dot{A}_2$ and $\dot{A}_3$. The space obtained from $A_1 \cup A_2 \cup A_3$ by identifying B to a point $[B]$ is called a *simple triod with centre at* $[B]$. Recall that by a *triod* we mean a continuum X which contains a subcontinuum C such that $X \setminus C$ is a union of three nonvoid separated sets. An arc $A \subset M$ is said to be *free* (in M) if $\mathring{A}$ is an open subset of M. By a *domain* we mean an open connected set.

LEMMA 1.2. *Let X be a nondegenerate regular continuum and let G be a nonvoid domain in X such that* $\operatorname{ord}_x X = 2$ *for each* $x \in G$. *Then* $\bar{G}$ *is either a simple closed curve or an arc.*

The proof can be easily supplied using the Whyburn theory of cyclic elements [5].

The next result gives the promised characterization of complete regularity.

THEOREM 1.3. *A continuum X is completely regular if and only if there exist a 0-dimensional compact subset F of X and a finite or countable null sequence of free arcs A_1, A_2, ... in X such that*

$$X = F \cup \bigcup \{A_n \colon n \geqslant 1\} \quad \textit{and} \quad A_j \cap F = \dot{A}_j$$

for each $j \geqslant 1$.

Remark. Observe that $\mathring{A}_i \cap \mathring{A}_j = \emptyset$ for $i \neq j$.

Proof. The sufficiency is trivial. Now, assume X is completely regular and consider the set $E_0 = \{x \in X \colon \operatorname{ord}_x X \geqslant 3\}$. Suppose there is an arc $L \subset E_0$ and consider an arbitrary point $y \in L$. Then $y = \lim x_n$, where $\operatorname{ord}_{x_n} X \geqslant 3$ for each $n \geqslant 1$. Since X is locally connected, by the "n-Bein-Satz" of Menger there is a simple triod T_n with centre at x_n for each $n \geqslant 1$. We may assume that $\operatorname{diam} T_n < 1/n$. Since $T_n \not\subset L$, it follows that $y \in \overline{X \setminus L}$. This implies that $\operatorname{Int} L = \emptyset$, a contradiction.

By Proposition 1.1 it follows that E_0 contains no nondegenerate subcontinuum and therefore $\dim E_0 \leqslant 0$. The set $E_1 = \{x \in X \colon \operatorname{ord}_x X = 1\} \setminus E_0$ is discrete and therefore countable. Moreover $\bar{E}_1 \setminus E_1 \subset E_0$. Hence $E_0 \cup E_1$ is compact of dimension $\leqslant 0$. If this set is void, then X is a point or a simple closed curve, and there is nothing to be proved. Hence we may assume that it is nonvoid. Then let G_1, G_2, ... denote all components of

$X\setminus(E_0 \cup E_1)$. From Proposition 1.1 and Lemma 1.2 it follows that $\bar{G}_n$ is either an arc or a simple closed curve. Moreover $\bar{G}_1, \bar{G}_2, \ldots$ form a null sequence by Proposition 1.1. Let E_2 be a set obtained by selecting one point from each G_n for which $\bar{G}_n$ is a simple closed curve. Then the set $F = E_0 \cup E_1 \cup E_2$ is compact and 0-dimensional. The reader can easily check that the closures of the components of the set $X\setminus F$ form a countable collection of arcs which together with F satisfy the requirements of the theorem. This completes the proof.

If M is compact, then by 2^M we denote the hyperspace of all closed subsets of M with the Vietoris topology. Hence the empty set is an isolated point of 2^M.

PROPOSITION 1.4. *Let $g\colon M \to N$ be a mapping between compact spaces. Then*

$$\{y \in N\colon\ g^{-1}\colon N \to 2^M \text{ is continuous at } y\}$$

is a dense G_δ-subset of N (see [4], p. 71).

All subcontinua of M form a subspace of 2^M denoted by $C(M)$. It is known that the topology of $C(M)$ inherited from 2^M coincides with the metric topology induced by the Hausdorff metric $\operatorname{dist}(\cdot,\cdot)$. A sequence $C_1, C_2, \ldots$ of subcontinua of M is said to *converge* to a continuum C_0; notation: $\{C_n\} \to C_0$, if $C_0, C_1, \ldots$ treated as points of $C(M)$ converge in the sense of Hausdorff metric. It is well known that if M is a continuum which is not hereditarily locally connected then there exists a convergence continuum C_0 in M, that is: C_0 is nondegenerate and there exists a sequence of pairwise disjoint continua $C_1, C_2, \ldots$ disjoint from C_0 such that $\{C_n\} \to C_0$.

PROPOSITION 1.5. *If X is a continuum which is not hereditarily locally connected then there exist a connected set $W \subset X$ and a nondegenerate continuum $C \subset X$ such that $C \subset \bar{W}\setminus W$.*

Proof. Let $A_0, A_1, \ldots$ be a sequence of nondegenerate pairwise disjoint continua such that $\{A_n\} \to A_0$. Let C' and C'' be two nondegenerate disjoint subcontinua of A_0. Applying [4], Theorem 7, p. 141, to the family $S = \{C', C'', A_1, A_2, \ldots\}$ we infer that there is a connected set $W \subset X$ containing infinitely many elements of S such that either $C' \cap W = \emptyset$ or $C'' \cap W = \emptyset$, say $C' \cap W = \emptyset$. Then $C = C'$ satisfies the conclusion of Proposition 1.5, which completes the argument.

By I we denote the unit interval $[0, 1]$.

LEMMA 1.6. *Let N be a compact space and let $\varphi\colon N \to I$ be a surjection such that $\varphi^{-1}(t)$ is a boundary subset of N for each $t \in I$. If P is a countable subset of I, then $\varphi^{-1}(I\setminus P)$ is a dense G_δ-subset of N.*

This follows from the Baire theorem.

One easily shows the following

PROPOSITION 1.7. *If there exists a monotone mapping from a compact space M onto a continuum, then M is a continuum.*

2. Dyer's theorem on monotone selectors. The aim of this section is to give a simple proof of an interesting result essentially established in a paper of Dyer [1] which however was not explicitly stated there. In this form this result appeared in [3] for the first time. All results from [3] heavily depend on this theorem.

THEOREM 2.1 (E. Dyer). *Let X and Y be nondegenerate continua and let $f\colon X \to Y$ be a monotone open surjection. Then there exists a dense G_δ-subset A of Y having the following property: for each $y \in A$, for each continuum $B \subset f^{-1}(y)$, for each $x \in \operatorname{Int}_{f^{-1}(y)} B$ and for each neighborhood U of B in X, there exist a continuum $Z \subset X$ containing B and a neighborhood V of y in Y such that $x \in \operatorname{Int} Z$, $(f|Z)^{-1}(V) \subset U$ and $f|Z\colon Z \to Y$ is a monotone surjection.*

Proof. Let $K(x, \varepsilon)$ denote the ε-ball about x in X. For each natural $n \geqslant 1$ let

$$M_n = \{(B, x) \in C(X) \times X\colon K(x, 1/n) \cap f^{-1}f(x) \subset B \subset f^{-1}f(x)\}.$$

Since f is open, M_n is compact. The map $f_n\colon M_n \to Y$ given by $f_n((B, x)) = f(x)$ is continuous, hence the set

$$A_n = \{y \in Y\colon f_n^{-1}\colon Y \to 2^{M_n} \text{ is continuous at } y\} \tag{1}$$

is a dense G_δ-subset of Y by Proposition 1.4. Then

$$A = \bigcap \{A_n\colon n \geqslant 1\}$$

is a dense G_δ-subset of Y. We shall show that A has the desired property. To this end fix a point $y_0 \in A$. Let $B_0 \subset f^{-1}(y_0)$ be a continuum, let x_0 be a point of $\operatorname{Int}_{f^{-1}(y_0)} B_0$ and let U_0 be a neighborhood of B_0 in X. There is an index $n \geqslant 1$ such that $K(x_0, 1/n) \cap f^{-1}(y_0) \subset B_0$. Since $f(x_0) = y_0$, $(B_0, x_0) \in M_n$. Thus

$$(B_0, x_0) \in f_n^{-1}(y_0). \tag{2}$$

Let F be a compact neighborhood of B_0 in X contained in U_0 and let

$$\langle F \rangle = \{D \in C(X)\colon D \subset F\}.$$

Let $\varepsilon < 1/2n$ be a positive real number. The set

$$G = M_n \cap [\langle F \rangle \times K(x_0, \varepsilon)]$$

is a neighborhood of (B_0, x_0) in M_n. Since $y_0 \in A_n$, there is an open neighborhood V of y_0 in Y such that

$$y \in V \Rightarrow f_n^{-1}(y) \cap G \neq \emptyset \quad \text{(see (1) and (2))}. \tag{3}$$

Consider a point $y \in V$. By (3) there is $(B, x) \in G$ such that $f_n((B, x)) = y$. It follows that $K(x, 1/n) \cap f^{-1}(y) \subset B \subset f^{-1}(y)$, $B \subset F$ and $x \in K(x_0, \varepsilon)$. Then

$$x \in \overline{K(x_0, \varepsilon)} \cap f^{-1}(y) \subset K(x, 1/n) \cap f^{-1}(y) \subset B \subset F \cap f^{-1}(y).$$

Hence for each $y \in V$ the set $\overline{K(x_0, \varepsilon)} \cap f^{-1}(y)$ is nonvoid and it is a subset of a single component of $F \cap f^{-1}(y)$; let this component be denoted by C_y. One easily sees that the set

$$Z = f^{-1}(Y \setminus V) \cup \bigcup \{C_y \colon y \in V\}$$

is compact; by Proposition 1.7 it is a continuum. This continuum and the set V satisfy the conclusion of the theorem, where U, B and x are replaced by U_0, B_0 and x_0, respectively.

COROLLARY 2.2. *Let X and Y be nondegenerate continua and let $f\colon X \to Y$ be a monotone open surjection with decomposable fibres. Then X contains a triod.*

Proof. From Theorem 2.1 we infer that there exist three different points $y_0, y_1, y_2 \in Y$ and a continuum C in X such that the map $f|C\colon C \to Y$ is a monotone surjection and $f^{-1}(y_i) \setminus C \neq \emptyset$. Then $C \cup \bigcup \{f^{-1}(y_j)\colon j = 0, 1, 2\}$ is the desired triod. This completes the proof.

By the Moore triodic theorem we obtain the following corollary to 2.2, which will be used as a step in a proof of a stronger theorem. S^2 is the 2-sphere.

LEMMA 2.3. *Let X be a continuum in S^2 and let f and Y be as in 2.2. Then Y is suslinian.*

3. Continuous decompositions of planar continua. Throughout this section X stands for a continuum in S^2 and $f\colon X \to Y$ is an open monotone mapping with decomposable fibres onto a nondegenerate continuum Y.

Using his theorem on monotone selectors E. Dyer proved in [2] that under the above assumptions Y must be hereditarily locally connected. In this section we give a shorter and conceptually simpler proof of this result. We obtain also some extra information on Y which, in the author's feeling, characterizes such continua Y. We shall derive all these facts from the following theorem. In spite of the fact that this theorem is not proved in [2], its proof involves several ideas used in that paper.

Notation. If N is a subcontinuum of Y and H is a subset of $S^2\setminus f^{-1}(N)$, then we denote

$$N(H) = \{y \in N\colon f^{-1}(y) \text{ is a limit of a sequence of continua from } H\}.$$

Obviously, $N(H)$ is a closed subset of N.

THEOREM 3.1. *If G is a component of $S^2\setminus X$, then* $\dim Y(G) \leqslant 0$.

Proof. Suppose $\dim Y(G) > 0$. Then there exists a nondegenerate continuum $Y' \subset Y(G)$ because $Y(G)$ is compact. Let G' denote the component of $S^2\setminus f^{-1}(Y')$ which contains G. There exist two points x_0, $x_1 \in f^{-1}(Y')$ accessible from G' such that $f(x_0) = y_0 \neq y_1 = f(x_1)$. Let $N \subset Y'$ be a continuum irreducible between y_0 and y_1. Let $M = f^{-1}(N)$ and let H denote the component of $S^2\setminus M$ containing G'. There is an arc L from x_0 to x_1 such that $\mathring{L} \subset H$. Then $H\setminus L$ is the union of two domains H_0 and H_1. Since $N(H) \subset N(H_0) \cup N(H_1) \cup \{y_0, y_1\}$ and by the construction $N(H) = N$, then for some $i = 0, 1$, the set $N(H_i)$ has nonvoid interior with respect to N, say $\mathrm{Int}_N N(H_0) \neq \emptyset$. It follows from Lemma 2.3 that N is hereditarily decomposable; hence there is a continuous surjection $\varphi\colon N \to I$ such that $\varphi^{-1}(t)$ is a boundary subcontinuum of N for each $t \in I$ (see [4], p. 216). Again by Lemma 2.3 there is a countable subset P of I containing 0 and 1 such that $\varphi^{-1}(t)$ is a one-point set for each $t \in I\setminus P$. It follows from Lemma 1.6 that $\varphi^{-1}(I\setminus P)$ is a dense G_δ-subset of N. Applying the theorem on monotone selectors to the map $f|M\colon M \to N$ we obtain a dense G_δ-subset A of N satisfying the conclusion of Theorem 2.1, where X, Y, f are replaced respectively by M, N and $f|M$. All these considerations lead to the conclusion that there is a point

$$\tilde{y} \in \mathrm{Int}_N N(H_0) \cap \varphi^{-1}(I\setminus P) \cap A.$$

Then $\tilde{y}$ has the following properties: (i) $\tilde{y}$ separates N between y_0 and y_1, (ii) $f^{-1}(\tilde{y}) \subset \mathrm{Fr}\, H_0$, (iii) $\tilde{y} \in A$. Now proceeding as in the proof of Theorem 2 in [2], p. 358, we get a contradiction. That technique of Dyer can also be adjusted to yield a construction of a skew curve of Kuratowski in S^2. This is another possibility of getting a contradiction. Thus the proof is completed.

LEMMA 3.2 (E. Dyer [2], Theorem 3). *Y is hereditarily locally connected.*

Proof. Suppose the theorem fails. By Proposition 1.5 there are a connected set $W \subset Y$ and a nondegenerate continuum $C \subset \overline{W}\setminus W$. Let V be the component of $S^2\setminus f^{-1}(C)$ containing the connected set $f^{-1}(W)$. Then $C = C(V)$, contrary to Theorem 3.1.

LEMMA 3.3. *Suppose $f^{-1}(Y\setminus\{y\})$ is contained in a single component of $S^2\setminus f^{-1}(y)$ for each $y \in Y$. Then Y is completely regular and embeds in S^2.*

Proof. Let D be an upper semi-continuous decomposition of S^2 in continua $f^{-1}(y)$ and the individual points of $S^2\setminus X$. Let q: $S^2 \to S^2/D$ be the projection. Then we may treat Y as the set $q(X)$ and f as the map determined by q. The space S^2/D is a Janiszewski space (see [4], Theorem, p. 506, and Theorem 9, p. 507). Our assumption about Y is equivalent to the fact that Y is a subset of a nondegenerate cyclic element E of S^2/D. It is known that E is topologically the sphere S^2 (see [4], Corollary 7, p. 533). According to Lemma 3.2 the lemma will be proved once we show that every arc in Y has nonvoid interior. Suppose, to the contrary, that there is an arc L in Y with void interior. Then $L \subset \overline{Y\setminus L}$. Also $Y\setminus L \subset E\setminus L$ and $E\setminus L$ is connected. There is a component G of $S^2\setminus q^{-1}(L)$ containing $q^{-1}(E\setminus L)$. Hence $f^{-1}(Y\setminus L) \subset G$ and from $L \subset \overline{Y\setminus L}$ we conclude that $L = L(G)$, contrary to Theorem 3.1.

If E is a cyclic element of a locally connected continuum, then no point of E separates E (see [4], Theorem 6, p. 313). Thus the above lemma applies to each cyclic element of Y. Hence we obtain the following

THEOREM 3.4. *Each cyclic element of Y is completely regular and embeds in S^2.*

COROLLARY 3.5. *Y is regular*(²).

Proof. By Theorem 3.4 and Proposition 1.1 each cyclic element of Y is regular. Since the property "of being regular" is extensible (see [4], Theorem 2, p. 325), Y is regular. This completes the proof.

4. An example and problems. Let $G_1, G_2, \ldots$ be the components of $S^2\setminus X$. It follows from Theorem 3.1 that $\bigcup\{Y(G_n)\colon n \geqslant 1\}$ does not contain a nondegenerate continuum because it is (an F_σ-set) of dimension $\leqslant 0$. One might wonder if the same is true for the set $Y(S^2\setminus X)$. Now we give an example violating this conjecture. Let N be a dendrite containing an arc Y with void interior in N (see, for instance, [4], p. 247, for such an example). According to a result in [2] there are a continuum $M \subset S^2$ and an open monotone surjection with decomposable fibres f: $M \to N$ (in fact M may be assumed to be the whole sphere S^2). Let $X = f^{-1}(Y)$. Then $Y(S^2\setminus X) = Y$ is a nondegenerate continuum.

The author believes that the answer to the following problem is affirmative.

PROBLEM 1. (**P 1288**) *Let Y be a continuum as in Theorem 3.4. Do there exist a continuum $X \subset S^2$ and an open monotone surjection f: $X \to Y$?*

Because of Theorem 3.4 the next problem is of an interest.

PROBLEM 2.(³) *Does there exist a universal continuum for the class of (planar) completely regular continua?*

(²) Perhaps this fact follows from the results in [2] but it is not proved there.

(³) S. D. Iliadis in his paper "Universal continuum for the class of completely regular continua" (preprint) has shown that there exists a universal planar completely regular continuum.

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