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COALGEBRA AND ALGEBRA STRUCTURE ON REPRESENTATIONS OF GENERAL COMPLEX LINEAR GROUPS

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In [2] Liulevicius proved in a "ridiculously simple" way that the graded abelian group $R = \bigoplus R_n$ such that R_{2n} is a complex representation ring $R(S_n)$ of a symmetric group S_n (with respect to $+$ and tensor multiplication $\times$) and $R_{2n+1} = 0$, with a product $\varphi\colon R\otimes R \to R$ induced by an operation of inducing a representation from $S_p \times S_q$ to S_{p+q} and with a coproduct $\psi\colon R \to R\otimes R$ induced by an operation of restriction of a representation from S_{p+q} to $S_p \times S_q$, is isomorphic to a Hopf algebra $C = Z[y_0, y_1, \ldots]$ with natural product of a polynomial ring on indeterminates $y_1, y_2, \ldots$ $(y_0 = 1)$ and a coproduct given by $\psi(y_n) = \sum_{p+q=n} y_p \otimes y_q$, grade $y_n = 2n$. The indeterminate y_n corresponds to a trivial, one-dimensional representation of S_n.

The purpose of the present paper is to give interpretations of some known facts concerning complex representations of general complex linear groups $G_n = GL(n, C)$ in terms of Hopf algebra structure on C. Unfortunately, a corresponding evenly graded abelian group $R(G) = \bigoplus_{n=0}^{\infty} R(G_n)$ with natural coproduct and with slightly less natural product is not a Hopf algebra, but its structure is completely determined by that of Hopf algebra C.

We use only those results on representations of symmetric groups which are proved in [2] and supplement them in Section 1 by two known results, proving them in style of [2]. In Section 2 we study a graded group $R(G)$ of complex representations of full general complex linear groups G_n; in Section 3 we determine the structure of coproduct. In Section 4 we define product and determine its structure. Analytic and antianalytic representations of G_n are considered in Section 5.

We preserve notation of [2]. All representations under consideration are complex, finite dimensional, continuous.

A Hopf algebra approach to the study of representations of general linear groups over finite fields is developed in [4].

1. Complementary results on representations of symmetric groups. The isomorphism $A: C \to R = \oplus R(S_n)$ mentioned above maps y_n onto a class of trivial one-dimensional representations of S_n. Schur inner product on $R(S_n)$ is denoted by $(\ ,\ )$ and is induced by a map which with a pair of representations U, V of S_n associates $\dim_C \mathrm{Hom}_{S_n}(U, V)$. It extends naturally to a pairing $R(S_p)\otimes R(S_q)\otimes R(S_p)\otimes R(S_q) \to Z$ and Frobenius reciprocity gives $(\varphi(a_p\otimes b_q), a'_{p+q}) = (a_p\otimes b_q, \psi(a'_{p+q}))$ for $a_p \in R(S_p)$, $b_q \in R(S_q)$, $a'_{p+q} \in R(S_{p+q})$. If $a, b \in C$ we write (a, b) for (Aa, Ab).

For each exponent sequence $E = (e_1, e_2, \ldots, e_s)$, where $e_1, \ldots, e_s$ are nonnegative integers we have a monomial $y^E = y_1^{e_1} \ldots y_s^{e_s}$ of degree $e_1 + 2e_2 + \ldots + se_s$. Let us remind that d_n denotes an element of C corresponding to a one-dimensional representation of S_n which sends $\sigma \in S_n$ onto $\mathrm{sgn}(\sigma)$.

LEMMA 1. $\sum\limits_{p+q=n} (-1)^p d_p y_q = 0.$

Proof. We use formulas: $(d_p, y_1^p) = (y_p, y_1^p) = 1$, $(d_p, y^F) = 0$ for $F \neq (p, 0, \ldots, 0)$ and $(y_p, y^F) = 1$ for all monomials y^F of degree p.

To compute $(d_p y_q, y^E) = (d_p\otimes y_q, \psi(y^E))$ for any monomial y^E of degree $p+q$ let us remark that by the above formulas we can omit all terms in $\psi(y^E)$ which are not of the form $y_1^k \otimes y^F$. The sum of remaining terms is

$$(y_1\otimes 1 + 1\otimes y_1)^{e_1} (y_1\otimes y_1 + 1\otimes y_2)^{e_2} \ldots (y_1\otimes y_{s-1} + 1\otimes y_s)^{e_s}$$

$$= \sum_{k_1=0}^{e_1} \cdots \sum_{k_s=0}^{e_s} \binom{e_1}{k_1} \cdots \binom{e_s}{k_s} (y_1\otimes 1)^{k_1} (1\otimes y_1)^{e_1-k_1} \ldots (y_1\otimes y_{s-1})^{k_s} (1\otimes y_s)^{e_s-k_s}$$

$$= \sum \cdots \sum \binom{e_1}{k_1} \cdots \binom{e_s}{k_s} y_1^{k_1+\ldots+k_s} \otimes y_1^{e_1-k_1} \ldots y_s^{e_s-k_s}.$$

Consequently

$$(d_p y_q, y^E) = \sum \cdots \sum \binom{e_1}{k_1} \cdots \binom{e_s}{k_s}$$

for all $k_1 + \ldots + k_s = p$ and the equality $\sum\limits_{k_j=0}^{e_j} (-1)^{k_j} \binom{e_j}{k_j} = 0$ implies the formula.

COROLLARY 2. *$d_n = \det(y_{j-i+1})_{i,j=1,\ldots,n} = W_n(y_1, \ldots, y_{n-1}) + (-1)^{n+1} y_n$ where $y_0 = 1$, $y_k = 0$ for $k < 0$ and W_n is a polynomial.*

In fact, elements d_n and the determinant above satisfy the same recurrence formula of Lemma 1.

COROLLARY 3. $y_n = \det(d_{j-i+1})_{i,j=1,\ldots,n}$.

Apply the automorphism D (see [2]), which interchanges d_k and y_k, to the above formula.

COROLLARY 4. *Elements $d_1, d_2, \ldots$ are algebraically independent in C and $Z[d_1, \ldots, d_n] = Z[y_1, \ldots, y_n]$ for all n.*

An exponent sequence $E = (e_1, \ldots, e_s)$ determines a partition $\pi(E) = \{\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_r\}$, $r = e_1 + \ldots + e_s$, of $n = e_1 + 2e_2 + \ldots + se_s$, in which i occurs precisely e_i times. A conjugate partition $\pi(E') = \pi'(E) = \{\lambda'_1 \geqslant \lambda'_2 \geqslant \ldots \geqslant \lambda'_s\}$ consists of $\lambda'_1 = e_1 + \ldots + e_s$, $\lambda'_2 = e_2 + \ldots + e_s, \ldots, \lambda'_s = e_s$. There is known a relation of partial ordering $\trianglelefteq$ between partitions of n: $\{\lambda_1, \lambda_2, \ldots\} \trianglelefteq \{\mu_1, \mu_2, \ldots\}$ iff $\lambda_1 + \ldots + \lambda_k \leqslant \mu_1 + \ldots + \mu_k$ for $k = 1, 2, \ldots$ (see [1]). It is easy to see that $\pi(E) \trianglelefteq \pi(F)$ implies $E \leqslant F$ (i.e. for $F = (f_1, \ldots, f_t)$ the first nonzero entry from the right in $(e_1 - f_1, e_2 - f_2, \ldots)$ is positive).

Using methods of [2] we improve Lemma 14 of [2] to

LEMMA 5. *If* $\langle c^E, y^F \rangle \neq 0$ *then* $\pi(E) \trianglelefteq \pi(F')$ *where* $c^E = c_1^{e_1} \ldots c_s^{e_s} = c_{\lambda_1} \ldots c_{\lambda_r}$ *and* c_i *is such an element of grade* $2i$ *in graded dual* C_* *of* C *that* c_i *vanishes on all monomials different from* y_1^i *and* $\langle c_i, y_1^i \rangle = 1$.

Proof. We have $\langle c^E, y^F \rangle = \langle c_{\lambda_1} \otimes \ldots \otimes c_{\lambda_r}, \psi^r(y^F) \rangle$, where $\psi^r\colon C \to C \otimes \ldots \otimes C$ (r times) is induced by $\psi\colon C \to C \otimes C$. The only term of the form $y_{i_1}^{a_1} \otimes \ldots \otimes y_{i_r}^{a_r}$ on which $c_{\lambda_1} \otimes \ldots \otimes c_{\lambda_r}$ does not vanish is $y_1^{\lambda_1} \otimes \ldots \otimes y_1^{\lambda_r}$.

For each function $\varepsilon\colon \{1, 2, \ldots, r\} \to \{0, 1\}$ let us denote $y(\varepsilon) = y_1^{\varepsilon(1)} \otimes \ldots \otimes y_1^{\varepsilon(r)} \in C \otimes \ldots \otimes C$ where $y_1^0 = 1$, $y_1^1 = y_1$. Let A_k, $k = 1, 2, \ldots, r$, be the set of all such functions ε that $|\varepsilon^{-1}(1)| = k$. Since

$$\psi^r(y^F) = (\psi^r(y_1))^{f_1} \ldots (\psi^r(y_t))^{f_t}$$

and $\psi^r(y_k) = \sum y_{k_1} \otimes \ldots \otimes y_{k_r}$, where $k_1 + \ldots + k_r = k$, then the terms in $\psi^r(y_k)$ which contain only y_1 and y_1^0 sum up to $\sum_{\varepsilon \in A_k} y(\varepsilon)$. Consequently the sum of such terms in $\psi^r(y^F)$ equals

$$\Big(\sum_{\varepsilon_1 \in A_1} y(\varepsilon_1)\Big)^{f_1} \ldots \Big(\sum_{\varepsilon_t \in A_t} y(\varepsilon_t)\Big)^{f_t} = \sum y(\varepsilon_{11}) \ldots y(\varepsilon_{1f_1}) \ldots y(\varepsilon_{t1}) \ldots y(\varepsilon_{tf_t})$$

where the sum extends over all $\varepsilon_{11}, \ldots, \varepsilon_{1f_1} \in A_1, \ldots, \varepsilon_{t1}, \ldots, \varepsilon_{tf_t} \in A_t$. Each choice of ε's determines such an arrangement of natural numbers (repetitions admitted) in a Young diagram of shape F (i.e. with f_k rows of length k) that a set $\varepsilon_{ij}^{-1}(1)$ is put in j-th row of length i in an increasing order. In this arrangement a number l may occur in first l columns only. It is clear that $y(\varepsilon_{11}) \ldots y(\varepsilon_{tf_t}) = y_1^{a_1} \otimes \ldots \otimes y_1^{a_r}$ where a_k is the number of entries of k in a diagram; then

$$a_1 + \ldots + a_k (\text{number of entries of } 1, 2, \ldots, k) \leqslant \mu'_1 + \ldots + \mu'_k$$

because $\mu'_1, \mu'_2, \ldots$ are lengths of columns of F.

The inequality $\langle c^E, y^F \rangle \neq 0$ implies that $y_1^{\lambda_1} \otimes \ldots \otimes y_1^{\lambda_r}$ occurs as some $y_1^{a_1} \otimes \ldots \otimes y_1^{a_r}$; then $\lambda_1 + \ldots + \lambda_k \leqslant \mu'_1 + \ldots + \mu'_k$, $k = 1, 2, \ldots$, i.e. $\pi(E) \trianglelefteq \pi(F')$.

In the same way as in [2] we conclude from Lemma 5 the well-known

COROLLARY 6. *For each partition* E *of* n, *elements* b_E *corresponding in* C

to irreducible representations $[\pi(E)]$ *of a symmetric group* S_n *satisfy*

$$b_E = y^E - \sum \langle Hb_F, y^E \rangle b_F \quad \text{where } \pi(E) \lhd \pi(F).$$

2. Complex polynomial representations of $GL(n, C)$. It is known that any polynomial representation of the group $G_n = GL(n, C)$ is isomorphic to

$$L_E(C^n) = \mathrm{Hom}_{S_e}([\pi(E)], (C^n)^{\otimes e})$$

where $E = (e_1, \ldots, e_s)$ is a partition of $e = e_1 + 2e_2 + \ldots + se_s$, $[\pi(E)]$ is an irreducible representation of S_e corresponding to E and $e_1 + \ldots + e_s \leqslant n$. The group S_e operates on $(C^n)^{\otimes e}$ by permuting components and G_n operates naturally on C^n. e (= degree of representation) and E are unique.

For each $n = 1, 2, \ldots$ let $R(G_n)$ be a ring (with respect to $+$ and tensor multiplication $\times$) of polynomial representations of G_n. Let v_n: $C \to R(G_n)$ be such an epimorphism of abelian groups that $v_n(b_E) = L_E(C^n)$. Thus for any finite dimensional representation V of S_e we have $v_n(A^{-1}(V)) = \mathrm{Hom}_{S_e}(V, (C^n)^{\otimes e})$.

Proposition 7. *Let* $a, b \in C$; *then* $v_n(ab) = v_n(a) \times v_n(b)$.

Proof. It is sufficient to prove the formula for monomials $a = y^E = y_1^{e_1} \ldots y_s^{e_s} = y_{\lambda_1} \ldots y_{\lambda_r}$, $b = y^F = y_1^{f_1} \ldots y_t^{f_t}$. We put $f = f_1 + \ldots + tf_t$, $m = e + f$ and let $S_E = S_{\lambda_1} \times \ldots \times S_{\lambda_r}$ be the Young subgroup of S_e. The product in C is determined by an operation of inducing representation, thus Ay^E is a representation of S_e induced from a trivial representation of S_E, i.e. $Ay^E = C[S_e] \otimes_{S_E} C$, and

$$v_n(y^E) = \mathrm{Hom}_{S_e}(C[S_e] \otimes_{S_E} C, (C^n)^{\otimes e}) = \mathrm{Hom}_{S_E}(C, (C^n)^{\otimes e}) = \{(C^n)^{\otimes e}\}^{S_E}.$$

Consequently

$$\begin{aligned} v_n(y^E y^F) &= v_n(A^{-1}(\mathrm{Ind}_{S_e \times S_f}^{S_m}(Ay^E \times Ay^F))) \\ &= \mathrm{Hom}_{S_m}(\mathrm{Ind}_{S_e \times S_f}^{S_m}(Ay^E \times Ay^F), (C^n)^{\otimes m}) \\ &= \mathrm{Hom}_{S_e \times S_f}(Ay^E \times Ay^F, (C^n)^{\otimes m}) \\ &= \mathrm{Hom}_{S_e \times S_f}((C[S_e] \otimes_{S_E} C) \times (C[S_f] \otimes_{S_F} C), (C^n)^{\otimes m}) \\ &= \mathrm{Hom}_{S_e \times S_f}(C[S_e \times S_f] \otimes_{S_E \times S_F} C, (C^n)^{\otimes m}) \\ &= \mathrm{Hom}_{S_E \times S_F}(C, (C^n)^{\otimes m}) \\ &= \{(C^n)^{\otimes (e+f)}\}^{S_E \times S_F} \\ &= \{(C^n)^{\otimes e}\}^{S_E} \otimes \{(C^n)^{\otimes f}\}^{S_F} \\ &= v_n(y^E) \times v_n(y^F). \end{aligned}$$

Corollary 8. *The kernel of* v_n *is an ideal of* C *generated (additively) by all such* b_E *that* $e_1 + \ldots + e_s > n$.

PROPOSITION 9. *The kernel of v_n is an ideal of C generated by d_{n+1}, $d_{n+2}, \ldots$ The ring $R(G_n)$ is isomorphic to a polynomial ring over Z generated by algebraically independent elements $v_n(d_k) = \bigwedge^k(C^n)$ for $k = 1, 2, \ldots, n$.*

Proof. A representation Ad_k is identical with $[\pi(k, 0, 0, \ldots)]$, then $d_k \in \operatorname{Ker} v_n$ for $k > n$. Conversely, let us assume that $b_E \in \operatorname{Ker} v_n$, then $e_1 + \ldots + e_s > n$. By the definition of b_E in [2] it follows that $D(b_{E'}) = b_E$. Since $b_{E'}$ is a linear form in y^F for $F \geqslant E'$, b_E is a linear form in d^F for $F \geqslant E'$. Let $F = (f_1, \ldots, f_t)$, $f_t > 0$; then $F \geqslant E'$ implies $t \geqslant e_1 + \ldots + e_s > n$, so d^F is divisible by d_t.

By Corollary 8 we can identify $R(G_n)$, as an abelian group, with an additive group of a cyclic C-module $C\gamma_n$, where annihilator of γ_n is $\operatorname{Ker} v_n = (d_{n+1}, d_{n+2}, \ldots)$. A generator γ_n corresponds to a one-dimensional trivial representation of G_n. By Proposition 7 we have $(ab)\gamma_n = a\gamma_n \times b\gamma_n$.

3. Restriction of representations and coproduct. Let p, q, n be non-negative integers and $p+q = n$. The group $G_p \times G_q$ is naturally embedded in G_n and any representation of G_n restricted to $G_p \times G_q$ may be uniquely decomposed into a direct sum of tensor product of irreducible representations of G_p and G_q. In this way we get maps $R(G_n) \to R(G_p) \otimes_Z R(G_q)$ and a coproduct

$$\psi_G \colon R(G) \to R(G) \otimes_Z R(G)$$

where $R(G) = \bigoplus_{n=0}^{\infty} R(G_n) = \bigoplus_{n=0}^{\infty} C\gamma_n$, G_0 is a trivial group and $R(G_n)$ has grade $2n$. We identify $a\gamma_p \otimes b\gamma_q$ with $(a \otimes b)\gamma_p \otimes \gamma_q$ and view it as a representation of the group $G_p \times G_q$.

THEOREM 10. *For any $a \in C$ we have*

$$\psi_G(a\gamma_n) = \psi(a) \sum_{p+q=n} \gamma_p \otimes \gamma_q, \qquad n = 1, 2, \ldots$$

Proof. It is sufficient to prove the formula for monomials $y^E = y_1^{e_1} \ldots y_s^{e_s}$. We proceed by induction on $e_1 + \ldots + e_s$.

Let $k \geqslant 1$; then $v_n(y_k)$ is k-th component $S_k(C^n)$ of a symmetric algebra on C^n with natural operation of G_n. Hence

$$\begin{aligned}
\psi_G(y_k \gamma_n) &= \sum_{p+q=n} \operatorname{Res}_{G_p \times G_q} S_k(C^n) \\
&= \sum \operatorname{Res}_{G_p \times G_q} S_k(C^p \oplus C^q) \\
&= \sum_{p+q=n} \sum_{i+j=k} S_i(C^p) \otimes S_j(C^q) \\
&= \sum \sum y_i \gamma_p \otimes y_j \gamma_q \\
&= \psi(y_k) \sum_{p+q=n} \gamma_p \otimes \gamma_q .
\end{aligned}$$

Assume that the formula holds for y^E and let $k \geqslant 1$. The coproduct ψ: $C \to C \otimes C$ is a ring homomorphism then using Proposition 7 we get

$$\begin{aligned}
\psi_G((y^E y_k)\gamma_n) &= \psi_G(y^E \gamma_n \times y_k \gamma_n) \\
&= \sum_{p+q=n} \mathrm{Res}_{G_p \times G_q}(y^E \gamma_n \times y_k \gamma_n) \\
&= \sum \mathrm{Res}_{G_p \times G_q}(y^E \gamma_n) \times \mathrm{Res}_{G_p \times G_q}(y_k \gamma_n) \\
&= \sum (\psi(y^E)\gamma_p \otimes \gamma_q) \times (\psi(y_k)(\gamma_p \otimes \gamma_q)) \\
&= \sum \psi(y^E)\psi(y_k)\gamma_p \otimes \gamma_q = \psi(y^E y_k) \sum_{p+q=n} \gamma_p \otimes \gamma_q
\end{aligned}$$

and the theorem follows.

COROLLARY 11. *We have* $\psi_G(\gamma_n) = \sum_{p+q=n} \gamma_p \otimes \gamma_q$ *and then for* $a \in C$

$$\psi_G(a\gamma_n) = \psi(a)\psi_G(\gamma_n).$$

COROLLARY 12. *The coproduct* ψ_G: $R(G) \to R(G) \otimes R(G)$ *determines on* $R(G)$ *a structure of coassociative, cocommutative graded coalgebra. It is completely determined by* C.

4. Product of representations. Let p, q, n be nonnegative integers and $p+q = n$. Groups $G_p \times G_q$ are not of finite index in G_n unless $p = 0$ or $q = 0$. Thus the usual algebraic construction of induced representation does not preserve finiteness of dimension. We avoid this difficulty using Frobenius reciprocity

$$\mathrm{Ind}_K^G(M) = \bigoplus (M, \mathrm{Res}_K^G(N))N$$

where $K \subset G$ are finite groups, N runs over simple G-modules, M is a finitely generated K-module, and $(,)$ is Schur inner product.

We define a product φ_G: $R(G) \otimes R(G) \to R(G)$ by the formula

$$\varphi_G(b_E \gamma_p \otimes b_F \gamma_q) = \sum_H (b_E \gamma_p \otimes b_F \gamma_q, \psi_G(b_H \gamma_{p+q}) b_H \gamma_{p+q}).$$

The sum is finite; in fact, for non-zero $b_E \gamma_p \in R(G_p)$, $b_F \gamma_q \in R(G_q)$ the coefficient at $b_H \gamma_{p+q}$ is equal to $(b_E \otimes b_F, \psi(b_H)) = (\varphi(b_E \otimes b_F), b_H)$, so vanishes for almost all H.

THEOREM 13. *For each* $a \in Z[y_1, \ldots, y_p] \subset C$, $b \in Z[y_1, \ldots, y_q] \subset C$ *we have*

$$\varphi_G(a\gamma_p \otimes b\gamma_q) = \varphi(a \otimes b)\gamma_{p+q}.$$

Proof. If a, b are homogeneous in C then $\varphi(a \otimes b) = \sum_H (\varphi(a \otimes b), b_H) b_H$ where $\deg b_H = \deg a + \deg b$ and the formula follows by additivity.

Remark that the formula in Theorem 13 is not valid for arbitrary a, b. For instance $d_{p+1}\gamma_p = 0$; then $\varphi_G(d_{p+1}\gamma_p \otimes d_q \gamma_q) = 0$ but $\varphi(d_{p+1} \otimes d_q)\gamma_{p+q} = d_{p+1} d_q \gamma_{p+q} \neq 0$.

COROLLARY 14. *The product φ_G: $R(G) \otimes R(G) \to R(G)$ determines on $R(G)$ a structure of associative, commutative graded algebra. It is completely determined by C.*

Unfortunately, product and coproduct structures on $R(G)$ are not so closely related as to determine on $R(G)$ a Hopf algebra structure. In fact, an easy computation shows that maps $\psi_G \varphi_G$, $(\varphi_G \otimes \varphi_G)(1 \otimes T \otimes 1)\psi_G \otimes \psi_G$ send an element $\gamma_p \otimes \gamma_q$, corresponding to a trivial representation of $G_p \times G_q$, onto $\sum_{r+t=p+q} \gamma_r \otimes \gamma_t$ and $\sum_{i+j=p} \sum_{k+l=q} \gamma_{i+k} \otimes \gamma_{j+l} = \sum_{r+t=p+q} m_r \gamma_r \otimes \gamma_t$ respectively, where m_r is a number of quadruples (i, j, k, l) of nonnegative integers such that $i+k = r$, $j+l = t$, $i+j = p$, $k+l = q$, i.e. a number of double cosets $S_r \times S_t \backslash S_{p+q}/S_p \times S_q$.

5. Analytic and antianalytic representations. A classification of all (complex) continuous, not necessarily polynomial, irreducible representations of groups G_n is well known (see [3], [5]). Any such representation is isomorphic to a tensor product

$$T(r_1, \ldots, r_n) \otimes \bar{T}(s_1, \ldots, s_n) \otimes \Delta(\zeta)$$

($r_1, \ldots, r_{n-1}, s_1, \ldots, s_{n-1}$ are nonnegative integers, r_n, s_n are arbitrary integers, ζ is a complex number) of an analytic representation $T(r_1, \ldots, r_n)$, of an antianalytic representation $\bar{T}(s_1, \ldots, s_n)$ and a one-dimensional representation $\Delta(\zeta)$ which sends $g \in G_n$ onto $|\det(g)|^{2\zeta} = \exp(\zeta \log|\det(g)|^2)$. Parameters $r_1, \ldots, s_n, \zeta$ and $r'_1, \ldots, s'_n, \zeta'$ correspond to isomorphic tensor products iff $r_i = r'_i$, $s_i = s'_i$, $i = 1, \ldots, n-1$, and $r_n - r'_n = s_n - s'_n = \zeta' - \zeta$.

An analytic representation $T(r_1, \ldots, r_n)$ is isomorphic to a tensor product of $T(r_1, \ldots, r_{n-1}, 0)$ and a one-dimensional representation $g \mapsto (\det g)^{r_n}$. If $r_n \geq 0$ then we easily identify $T(r_1, \ldots, r_n)$ as a representation $b_H \gamma_n$, where H corresponds to a partition $\{r_1 + \ldots + r_n, \ldots, r_n\}$. Similarly, a representation $\bar{T}(s_1, \ldots, s_n)$ is isomorphic to a tensor product of $\bar{T}(s_1, \ldots, s_{n-1}, 0)$ and a one-dimensional representation $g \mapsto \overline{(\det g)^{s_n}}$; if $s_n \geq 0$ then we easily identify $\bar{T}(s_1, \ldots, s_n)$ as a complex conjugate of $b_F \gamma_n$ for F corresponding to a partition $\{s_1 + \ldots + s_n, \ldots, s_n\}$.

To describe the ring of classes of continuous representations of general linear groups let us denote for each natural number n by $\dot{D}_n$ the group ring $Z[\{d_n^\zeta\}_{\zeta \in C}]$ of the additive group of C, i.e. $d_n^\zeta d_n^{\zeta'} = d_n^{\zeta+\zeta'}$; clearly $D_1 = D_2 = \ldots$ Moreover we introduce new generators $\bar{\gamma}_n$ $(n = 1, 2, \ldots)$ of cyclic C-modules $C\bar{\gamma}_n$ such that $\mathrm{Ann}\, \bar{\gamma}_n = \mathrm{Ann}\, \gamma_n$ and we identify the ring of isomorphism classes of antianalytic representations of G_n with $C\bar{\gamma}_n$.

THEOREM 15. *The ring $\tilde{R}(G_n)$ of equivalence classes of all continuous representations of a group G_n (with respect to $+$ and $\times$) is isomorphic to a factor ring of the ring $C\gamma_n \otimes_Z C\bar{\gamma}_n \otimes_Z D_n$ by the ideal generated by the element $d_n \gamma_n \otimes d_n \bar{\gamma}_n \otimes 1 - \gamma_n \otimes \bar{\gamma}_n \otimes d_n^1$.*

A coproduct structure on $\tilde{R}(G) = \bigoplus_{n=0}^{\infty} \tilde{R}(G_n)$ induced by restriction is determined by maps $\tilde{\psi}_{pq}$: $\tilde{R}(G_n) \to \tilde{R}(G_p) \otimes \tilde{R}(G_q)$ for $n = p+q$ which satisfy the conditions

(i) *$\tilde{\psi}_{pq}$ is a ring homomorphism;*

(ii) *$\tilde{\psi}_{pq}$ extends restrictions $C\gamma_n \to C\gamma_p \otimes C\gamma_q$ and the corresponding restrictions $C\bar{\gamma}_n \to C\bar{\gamma}_p \otimes C\bar{\gamma}_q$;*

(iii) *$\tilde{\psi}_{pq}(d_n^\zeta) = d_p^\zeta \otimes d_q^\zeta$.*

The proof is just a verification.

There is no natural extension of product structure on $R(G)$ to such a structure on $\tilde{R}(G)$. In fact, for $a\gamma_p \in C\gamma_p$, $b\gamma_q \in C\gamma_q$ as in Theorem 13 we have $\varphi_G(d_p a\gamma_p \otimes b\gamma_q) = (d_p ab)\gamma_{p+q}$. Multiplication by d_p is an invertible map in $\tilde{R}(G_p)$ but d_p-divisible part of $\tilde{R}(G_n)$ is zero for $n > p$.

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