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REMARKS ON SIMILARITY AND QUASISIMILARITY OF OPERATORS

BY

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The paper is divided into two parts. The first part concerns an extension of Friedrichs' method of similarity. The second one is devoted to quasisimilar subnormal operators.

1. Let $L(E)$ be the Banach algebra of all bounded linear operators in a Banach space E. Assume we are given an operator $T \in L(E)$, a second Banach space W, and a continuous linear map $\varphi\colon w \in W \to \varphi w \in L(E)$ (with $\ker\varphi = 0$). Let $\Gamma\colon W \to L(E)$ be a continuous map such that

(1) $$T\Gamma(w) - \Gamma(w)T = \varphi w, \qquad w \in W.$$

We would like to know for which $w \in W$ operators T and $T + \varphi w$ are similar. We use the method developed by Friedrichs [3]. In order to do this we also assume that there is a continuous map $\psi\colon W \times W \to W$ such that

(2) $$\varphi\psi(w_1, w_2) = \Gamma(w_1)\varphi w_2, \qquad w_1, w_2 \in W.$$

Suppose additionally that $\psi(\cdot, w_0)$ satisfies, for every $w_i \in W$ $(i = 0, 1, 2)$ and $\|w_i\| \leqslant 1$ $(i = 1, 2)$ the following condition (the local Lipschitz condition):

(3) $$\|\psi(w_1, w_0) - \psi(w_2, w_0)\| \leqslant k\|w_0\|\,\|w_1 - w_2\|.$$

Now applying the method of Friedrichs we have

PROPOSITION 1. *Let $\varphi\colon W \to L(E)$ be a continuous linear map with zero kernel. Suppose we are given an operator $T \in L(E)$ and a continuous map $\Gamma\colon W \to L(E)$ with $\Gamma(0) = 0$ which satisfies* (1). *Assume that there is a continuous map $\psi\colon W \times W \to W$ which satisfies* (2) *and* (3). *Assume that $\|z\| < \delta$ implies $\|\Gamma(z)\| < 1$ for a certain $\delta > 0$. If*

$$\frac{\|w\|}{1 - \|w\|k} < \delta,$$

then the operators $T + \varphi w$ and T are similar.

Proof. We want to find an invertible $U \in L(E)$ such that

$$U(T+\varphi w) = TU.$$

Following Friedrichs we put $U = I+\Gamma(z)$. Hence

$$(I+\Gamma(z))(T+\varphi w) = T(I+\Gamma(z))$$

is equivalent to

$$\varphi z = \Gamma(z)\varphi w+\varphi w.$$

By (2) we can write

$$\varphi z-\varphi\psi(z, w) = \varphi(z-\psi(z, w)) = \varphi w,$$

and so

$$z-\psi(z, w) = w. \tag{4}$$

Now it is clear (by (3)) that for any $w \in W$ ($w \neq 0$) such that

$$\frac{\|w\|}{1-\|w\|k} < \delta$$

the sequence $\{u_n\} \subset W$ defined by

$$u_0 = w,$$
$$u_1 = \psi(u_0, w)+w,$$
$$\dots\dots\dots\dots$$
$$u_{n+1} = \psi(u_n, w)+w$$

is convergent to the unique solution z_0 of (4). Moreover, by our choice of w and δ we have

$$\|z_0\| \leqslant \|w\|+k\|z_0\|\|w\|.$$

It follows that

$$\|z_0\| \leqslant \frac{\|w\|}{1-\|w\|k} < \delta,$$

and so $\|\Gamma(z_0)\| < 1$. Thus the operator $I+\Gamma(z_0)$ is invertible, and the proof is complete.

Now we give some illustrating examples.

EXAMPLE 1. Let $T \in L(E)$. Assume that for certain $X_1, X_2 \in L(E)$ we have
(a) $TX_1-X_1T = [T, X_1]$ is a one-dimensional operator C_1;
(b) X_2 commutes with T;
(c) if $C_1 z = (z, w)\mathring{x}$, then $X_s\mathring{x} = \lambda_s\mathring{x}$, $s = 1, 2$.

Put

$$\Gamma(\alpha) = \alpha X_1 + \alpha^2 X_2, \quad \alpha \in C, \ \varphi\alpha = [T, \Gamma(\alpha)] = \alpha C_1.$$

Now

$$\Gamma(\alpha)\varphi\beta = \varphi\psi(\alpha, \beta), \quad \text{where } \psi(\alpha, \beta) = (\alpha\lambda_1 + \alpha^2\lambda_2)\beta.$$

Since $\psi(\alpha, \beta)$ satisfies (3) (with $k = |\lambda_1| + 2|\lambda_2|$) and (2), Proposition 1 can be applied.

EXAMPLE 2. Let $T, X \in L(E)$ be a pair of operators in E such that $X[T, X] = [T, X]X = 0$. Now put

$$\Gamma(\alpha) = \alpha X, \quad \varphi\alpha = \alpha[T, X], \ \alpha \in C.$$

We have

$$\Gamma(\alpha)\varphi\beta = \alpha\beta[T, X]X = 0.$$

Take any α such that $|\alpha|\,\|X\| < 1$. Then the operators $T + \varphi\alpha$ and T are similar.

We can specify the above general assumptions in a special case. Namely, if $E = H$ is a Hilbert space and S is a quasinormal operator ($SS^*S = S^*SS$), then we can simply put $T = S^*$ and $X = S$, for they satisfy the above assumptions.

Remark 1. Let

$$T = \begin{pmatrix} S & A \\ 0 & B \end{pmatrix}, \quad X = \begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix}$$

be a pair of operators on E (where the above matrices are written with respect to an arbitrary decomposition $E = E_1 \oplus E_2$).

Since $X[T, X] = 0$, we obtain the similarity of $T + [T, X]$ and T.

In fact, $I + X$ is invertible and

$$(I + X)(T + [T, X]) = T(I + X).$$

This remark suggests that the method of Friedrichs can also be applied in other situations.

2. Now we shall consider quasisimilar subnormal operators in a complex Hilbert space. Let us recall this notion.

We say that $T_s \in L(H_s)$, $s = 1, 2$, are *quasisimilar* if there are linear, bounded mappings $X_1: H_2 \to H_1$ and $X_2: H_1 \to H_2$ with the following properties:

$$\ker X_i = \{0\}, \quad \overline{R(X_i)} = H_i \quad \text{and} \quad X_2 T_1 = T_2 X_1, \quad T_1 X_1 = X_1 T_2.$$

In his work [2], Clary has proved that quasisimilar subnormal operators have equal spectra. As we shall see below they also have equal essential spectra under some additional conditions.

Let $\sigma_e(T)$ denote the essential spectrum of $T \in L(E)$. We have the following

PROPOSITION 2. *Assume that $T_s \in L(H_s)$ are quasisimilar subnormal operators with minimal normal extensions $N_s \in L(K_s)$, $s = 1, 2$. Assume that $\sigma(N_s) \subset \partial\sigma(T_s)$. Then $\sigma_e(T_1) = \sigma_e(T_2)$.*

Proof. By the above-mentioned result of Clary, $\sigma(T_1) = \sigma(T_2)$. Thus $\sigma(N_1) = \partial\sigma(T_1) = \partial\sigma(T_2) = \sigma(N_2)$. By symmetry it is enough to check that $\lambda \notin \sigma_e(T_1)$ implies $\lambda \notin \sigma_2(T_2)$. Let $\lambda \notin \sigma_e(T_1)$. We have two cases: $\lambda \notin \sigma(N_1)$ and $\lambda \in \sigma(N_1)$.

If $\lambda \notin \sigma(N_1)$, then $\lambda \notin \sigma(N_2)$. Hence $R(\lambda - T_2)$ is closed. But

$$\dim\ker(\lambda - T_2) = \dim\ker(\lambda - T_1) < \infty$$

and

$$\dim\ker(\lambda - T_2)^* = \dim\ker(\lambda - T_1)^* < \infty,$$

so $\lambda \notin \sigma_e(T_2)$.

If $\lambda \in \sigma(N_1)$, then $\lambda \in \partial\sigma(T_1)$. By the result of Putnam [4], either λ is an isolated point of the point spectrum of T_1 ($\sigma_p(T_1)$) with a finite multiplicity or $\lambda \in \sigma_e(T_1)$. Since the latter is impossible, we have $\lambda \in \sigma_p(T_1)$. Hence $\lambda \in \sigma_p(T_2)$, and so $\lambda \notin \sigma_e(T_2)$. The proof is complete.

The assumptions of Proposition 2 are not necessary for the equality of essential spectra of quasisimilar subnormal operators, as the following example shows.

EXAMPLE 3. Let A^2 denote the Bergman space in the unit disc D. Denote by B_z the operator of multiplication by z on A^2. Let S be a subnormal operator quasisimilar to B_z. Then $\sigma_e(S) = \sigma_e(B_z) = \partial D$.

Indeed, it is clear that S is cyclic. Thus there exists a measure μ ($\operatorname{supp}\mu \subseteq \bar{D}$) such that S is unitarily equivalent to the operator T_μ of multiplication by z on $H^2(\mu)$. By our assumption there exists $X\colon H^2(\mu) \to A^2$ such that $XT_\mu = B_z X$. It follows that $Xp = \phi p$ for every polynomial p, where $\phi = X1$. Hence

$$\int |p\phi|^2\, dA \leqslant \|X\|^2 \int |p|^2\, d\mu,$$

where dA denotes two-dimensional Lebesgue measure. Since ϕ is cyclic for B_z, $\phi(0) \neq 0$. Thus

$$\int_D \log|\phi|^2\, dA = 2\int_0^1\Big(\int_0^{2\pi} \log|\phi(re^{i\theta})|^2\, d\theta\Big) r\,dr = 2\pi \log|\phi(0)| > -\infty.$$

Now applying the reasoning of Brennan in [1], p. 175, we have

$$|p(\lambda)|^2 \leqslant C_\lambda \int |p|^2 |\phi|^2 dA \leqslant C_\lambda \|X\|^2 \int |p|^2 d\mu.$$

It follows that $(\lambda - T_\mu) H^2(\mu)$ is closed in $H^2(\mu)$. Since

$$\dim\ker(\lambda - T_\mu)^* = \dim\ker(\lambda - B_z)^* = 1,$$

$\lambda - T_\mu$ is Fredholm. But $\sigma(T_\mu) = \bar{D}$, whence $\partial D \subseteq \sigma_e(T_\mu)$. Consequently,

$$\sigma_e(S) = \sigma_e(T_\mu) = \partial D = \sigma_e(B_z).$$

By the way let us note the following simple corollary to the above-mentioned result of Clary. But first recall the notation. For $T \in L(H)$ we denote by $\operatorname{Rat} T$ the algebra of operators of the form $r(T)$, where r is a rational function with poles off the spectrum $\sigma(T)$.

COROLLARY. *Assume that $T \in L(H)$ and $S \in L(K)$ are hyponormal and quasisimilar ($XT = SX$ and $TY = YS$). If there is a finite number of vectors $f_1, \ldots, f_n \in H$ for which*

$$\bigvee_i \operatorname{Rat}(T) f_i = H$$

($\bigvee_i M_i$ denotes the linear span of M_i), then

$$\bigvee_i \operatorname{Rat}(S) X f_i = K.$$

Proof. Let $y \perp r(S) X f_i$ for every $r \in \operatorname{Rat} \sigma(S)$, $i = 1, \ldots, n$. Since $\sigma(T) = \sigma(S)$, $r \in \operatorname{Rat} \sigma(T)$ and we have

$$0 = (y, r(S) X f_i) = (y, Xr(T) f_i),$$

and so $(y, Xf) = 0$, $f \in H$. Hence $y = 0$. The proof is complete.

Remark 2. Let $\mathscr{K}(H)$ stand for the ideal of compact operators in a Hilbert space H. Denote by π the projection onto the Calkin algebra:

$$\pi\colon L(H) \to L(H)/\mathscr{K}(H).$$

Suppose S and T are given hyponormal operators on H which are similar modulo compact, i.e.,

$$S = X^{-1} T X + K, \qquad K \in \mathscr{K}(H) = \mathscr{K}.$$

Assume that $[T^*, T] \in \mathscr{K}$. Then $[S^*, S] \in \mathscr{K}$.

In fact, $\pi(T)$ is normal in $L(H)/\mathscr{K}$ and

$$\pi(S) = \pi(X)^{-1} \pi(T) \pi(X).$$

Hence, applying Corollary 1 of [5], we see that $\pi(S)$ is also normal, and so $[S^*, S] \in \mathscr{K}$.

Remark 3. If S and T are quasisimilar and quasinormal (see Example 2) and $[S^*, S] \in \mathscr{K}$, then $[T^*, T] \in \mathscr{K}$.

This is immediate by [7].

PROBLEM (P 1345). In view of the above remarks (and Corollary 1) we ask whether for quasisimilar hyponormal operators S and T the compactness of $[S^*, S]$ implies the compactness of $[T^*, T]$.

Now we shall give an application of the above-mentioned result of Clary to subnormal operators. We say that a collection $A_1, \ldots, A_n$ of commuting operators on H has a *commuting normal extension* if there exist commuting normal operators $N_1, \ldots, N_n$ defined on some $K \supset H$ with

$$A_i f = N_i f, \quad f \in H, \; i = 1, \ldots, n.$$

In what follows $\sigma(A_1, \ldots, A_n)$ stands for the Taylor joint spectrum of $A_1, \ldots, A_n$ (see [6]). Denote by $\mathscr{P}(A_1, \ldots, A_n)$ the smallest Banach algebra with unit generated by $A_1, \ldots, A_n$.

PROPOSITION 3. *Suppose we are given two collections $\{A_1, \ldots, A_n\}$ (on H_1) and $\{B_1, \ldots, B_n\}$ (on H_2) of commuting subnormal operators with normal extensions on larger spaces. If $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_n\}$ are quasisimilar ($XA_i = B_i X$, $A_i Y = YB_i$, $i = 1, \ldots, n$), then*

$$\sigma(A_1, \ldots, A_n)^{\wedge} = \sigma(B_1, \ldots, B_n)^{\wedge},$$

where $Z^{\wedge}$ denotes the polynomial convex hull of a compact set $Z \subset C^n$.

Proof. For $T \in L(H)$ we denote by $r(T)$ the spectral radius of T. By symmetry it is enough to prove the implication

$$\lambda \notin \sigma(A_1, \ldots, A_n)^{\wedge} \Rightarrow \lambda \notin \sigma(B_1, \ldots, B_n)^{\wedge}.$$

Let $\sigma_{\mathscr{P}(A_1, \ldots, A_n)}(A_1, \ldots, A_n)$ denote the joint spectrum of $(A_1, \ldots, A_n)$ with respect to $\mathscr{P}(A_1, \ldots, A_n) = \mathscr{P}$. Then by Theorem 5.2 of [6] we have

$$\sigma(A_1, \ldots, A_n)^{\wedge} = \sigma_{\mathscr{P}}(A_1, \ldots, A_n).$$

Hence there exist $S_1, \ldots, S_n \in \mathscr{P}$ such that

$$\sum_{i=1}^{n} S_i(\lambda_i - A_i) = I.$$

Choose sequences of elements $p_{ki}(A_1, \ldots, A_n) \in \mathscr{P}$ such that

$$\lim_k \|p_{ki}(A_1, \ldots, A_n) - S_i\| = 0, \qquad i = 1, \ldots, n.$$

By the result of Clary we have

$$\sigma\big(p_{ki}(B_1, \ldots, B_n) - p_{li}(B_1, \ldots, B_n)\big) = \sigma\big(p_{ki}(A_1, \ldots, A_n) - p_{li}(A_1, \ldots, A_n)\big)$$

for each k, l and $i = 1, \ldots, n$. We can write

$$\begin{aligned}\|p_{ki}(B_1, \ldots, B_n) - p_{li}(B_1, \ldots, B_n)\| &= r\big(p_{ki}(B_1, \ldots, B_n) - p_{li}(B_1, \ldots, B_n)\big)\\ &= r\big(p_{ki}(A_1, \ldots, A_n) - p_{li}(A_1, \ldots, A_n)\big)\\ &= \|p_{ki}(A_1, \ldots, A_n) - p_{li}(A_1, \ldots, A_n)\|.\end{aligned}$$

Hence

$$\lim_k p_{ki}(B_1, \ldots, B_n) = R_i \quad \text{for a certain } R_i \in \mathscr{P}(B_1, \ldots, B_n).$$

Thus

$$X = X \sum_{i=1}^{n} S_i(\lambda_i - A_i) = \sum_{i=1}^{n} R_i(\lambda_i - B_i) X.$$

But $\overline{R(X)} = H_2$, so

$$\sum_{i=1}^{n} R_i(\lambda_i - B_i) = I, \quad \text{i.e.,} \quad \lambda \notin \sigma(B_1, \ldots, B_n)^{\wedge}.$$

The proof is complete.

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