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ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF DARBOUX PROBLEM FOR PARTIAL DIFFERENTIAL-FUNCTIONAL EQUATIONS

BY

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In this paper we consider a partial differential equation

$$u_{xy}(x,y) = F\Big((x,y,u\big(\gamma_1(x,y),\delta_1(x,y)\big),u_x\big(\gamma_2(x,y),\delta_2(x,y)\big), u_y\big(\gamma_3(x,y),\delta_3(x,y)\big),u_{xy}\big(a(x,y),\beta(x,y)\big)\Big) \tag{1}$$

with the boundary conditions

$$\begin{cases} u(x,0) = \sigma(x) & \text{for } 0 \leqslant x \leqslant a, \\ u(0,y) = \tau(y) & \text{for } 0 \leqslant y \leqslant b, \end{cases} \tag{2}$$

where $\sigma\colon [0,a] \to R^1$ and $\tau\colon [0,b] \to R^1$ are functions of class C^1 satisfying $\sigma(0) = \tau(0)$.

We shall be interested only in solutions u which are continuous in the rectangle $\varDelta = [0,a]\times[0,b]$ together with its partial derivatives u_x, u_y, u_{xy}. The set of all such functions will be denoted by $C^*(\varDelta)$. The problem consisting in finding a solution of equation (1) fulfilling conditions (2) will be called the *Darboux problem*.

The Darboux problem for equation (1) with $\gamma_i(x,y) = x$, $\delta_i(x,y) = y$, where $i = 1,2,3$, and F not depending on the last variable, was considered by many authors under various conditions; for a more detailed information and bibliography see [2], [4] and [6].

In this paper we shall prove the existence and uniqueness of solutions of the Darboux problem by the method of successive approximations. We shall also give estimations of the error and a theorem stating continuous dependence of solutions on the right-hand side of equation (1). Our results are generalizations of those in [5].

More precisely, we shall deal with the following case:

$$0 \leqslant \gamma_i(x,y) \leqslant x,\ 0 \leqslant \delta_i(x,y) \leqslant y,\ 0 \leqslant a(x,y) \leqslant kx,\ 0 \leqslant \beta(x,y) \leqslant ly,$$
$$k, l \in [0,1],\ i = 1,2,3.$$

Essential in this case is that theorems established here involve some relations between the Lipschitz coefficient of the function F with respect to the last variable and the functions α and β. All our results are obtained by using the general idea of Ważewski [7] (see also [1] and [3]).

If a function $u \in C^*(\Delta)$ is a solution of the Darboux problem (1)-(2), then, for the function z continuous on Δ and defined by the formula $z(x, y) = u_{xy}(x, y)$, we have

$$u(x, y) = \sigma(x)+\tau(y)-\sigma(0)+\int_0^x\int_0^y z(u, v)\,du\,dv, \tag{3}$$

$$u_x(x, y) = \sigma'(x)+\int_0^y z(x, v)\,dv,$$

$$u_y(x, y) = \tau'(y)+\int_0^x z(u, y)\,du.$$

Consequently, the function z fulfils the equation

$$\begin{aligned} z(x, y) = F\Big(x, y, &\ \sigma\big(\gamma_1(x, y)\big)+\tau\big(\delta_1(x, y)\big)-\sigma(0)+ \\ &+\int_0^{\gamma_1(x,y)}\int_0^{\delta_1(x,y)} z(u, v)\,du\,dv,\ \sigma'\big(\gamma_2(x, y)\big)+\int_0^{\delta_2(x,y)} z\big(\gamma_2(x, y), v\big)\,dv, \\ &\tau'\big(\delta_3(x, y)\big)+\int_0^{\gamma_3(x,y)} z\big(u, \delta_3(x, y)\big)\,du,\ z\big(\alpha(x, y), \beta(x, y)\big)\Big). \end{aligned} \tag{4}$$

Conversely, if a function z, continuous on Δ, fulfils (4), then the function $u \in C^*(\Delta)$ defined by (3) is a solution of the Darboux problem (1)-(2). Thus the Darboux problem (1)-(2) is equivalent to the problem of solving the integral equation (4).

Putting in equation (4)

$$\begin{aligned} f(x, y, z, p, q, r) = F\Big(x, y, &\ \sigma\big(\gamma_1(x, y)\big)+\tau\big(\delta_1(x, y)\big)-\sigma(0)+z, \\ &\sigma'\big(\gamma_2(x, y)\big)+p,\ \tau'\big(\delta_3(x, y)\big)+q,\ r\Big), \end{aligned}$$

we get an integral equation of the form

$$\begin{aligned} z(x, y) = f\Big(x, y, &\int_0^{\gamma_1(x,y)}\int_0^{\delta_1(x,y)} z(u, v)\,du\,dv,\ \int_0^{\delta_2(x,y)} z\big(\gamma_2(x, y), v\big)\,dv, \\ &\int_0^{\gamma_3(x,y)} z\big(u, \delta_3(x, y)\big)\,du,\ z\big(\alpha(x, y), \beta(x, y)\big)\Big), \end{aligned} \tag{5}$$

with which we shall deal.

Remark 1. If $\gamma_i(0, 0) = \delta_i(0, 0) = \alpha(0, 0) = \beta(0, 0) = 0$ for $i = 1, 2, 3$, then we assume in the sequel that also $f(0, 0, \ldots, 0) = 0$ and

$z(0,0) = 0$. The general case can be reduced to that one by the substitution $z(x, y) - Axy = z^*(x, y)$, where A is a solution of the equation

$$z = f(0, 0, 0, 0, 0, z),$$

(it is supposed that there exists a solution of this equation).

1. Assumptions and lemmas.

Assumption H_1. Suppose that

1° $f\colon \Delta \times R^4 \to R^1$ is continuous ($\Delta = [0, a] \times [0, b]$);

2° functions $\gamma_i, \alpha\colon \Delta \to [0, a]$ and $\delta_i, \beta\colon \Delta \to [0, b]$, $i = 1, 2, 3$, are continuous on Δ;

3° there exists a non-negative and continuous function

$$\omega\colon \Delta \times R^4_+ \to R^1_+ \qquad (x = (x_1, \ldots, x_n) \in R^n_+ \Leftrightarrow x \in R^n \text{ and } x_i \geqslant 0),$$

non-decreasing with respect to the last four variables z, p, q, r, and which fulfils the condition

$$\omega(x, y, 0, 0, 0, 0) \equiv 0;$$

moreover, for any $(x, y, z_i, p_i, q_i, r_i) \in \Delta \times R^4$, $i = 1, 2$, we have the inequality

$$(6) \quad |f(x, y, z_1, p_1, q_1, r_1) - f(x, y, z_2, p_2, q_2, r_2)| \leqslant \omega(x, y, |z_1 - z_2|, |p_1 - p_2|, |q_1 - q_2|, |r_1 - r_2|).$$

Assumption H_2. Suppose that

1° there exists a non-negative and continuous function $\bar{g}\colon \Delta \to R^1_+$ being a solution of the inequality

$$(7) \quad \omega\Big(x, y, \int_0^{\gamma_1(x,y)} \int_0^{\delta_1(x,y)} g(u, v)\,du\,dv, \int_0^{\delta_2(x,y)} g\big(\gamma_2(x, y), v\big)\,dv, \int_0^{\gamma_3(x,y)} g\big(u, \delta_3(x, y)\big)\,du, g\big(\alpha(x, y), \beta(x, y)\big)\Big) + h(x, y) \leqslant g(x, y),$$

where

$$h(x, y) = \sup_{0 \leqslant \xi \leqslant x} \sup_{0 \leqslant \eta \leqslant y} |f(\xi, \eta, 0, 0, 0, 0)|;$$

2° in the class of functions satisfying the condition

$$0 \leqslant g(x, y) \leqslant \bar{g}(x, y), \qquad (x, y) \in \Delta,$$

the function g, $g(x, y) \equiv 0$ for $(x, y) \in \Delta$, is the only measurable solution of the equation

$$(8) \quad g(x, y) = \omega\Big(x, y, \int_0^{\gamma_1(x,y)} \int_0^{\delta_1(x,y)} g(u, v)\,du\,dv, \int_0^{\delta_2(x,y)} g\big(\gamma_2(x, y), v\big)\,dv, \int_0^{\gamma_3(x,y)} g\big(u, \delta_3(x, y)\big)\,du, g\big(\alpha(x, y), \beta(x, y)\big)\Big).$$

Remark 2. It is easy to prove that conditions 1° and 2° of H_2 are fulfilled if inequality (7) has the form

$$K_1 \int_0^{\gamma_1(x,y)} \int_0^{\delta_1(x,y)} g(u,v)\,du\,dv + K_2 \int_0^{\delta_2(x,y)} g\big(\gamma_2(x,y), v\big)\,dv +$$
$$+K_3 \int_0^{\gamma_3(x,y)} g\big(u, \delta_3(x,y)\big)\,du + K_4 g\big(\alpha(x,y), \beta(x,y)\big) + h(x,y) \leqslant g(x,y),$$

where K_i, $i = 1, 2, 3, 4$, are non-negative constants and

$$K_1 ab + K_2 b + K_3 a + K_4 < 1.$$

Assumption H_3. Suppose that

1° there exists a function $w\colon \Delta \to R_+^1$, non-negative, continuous and non-decreasing with respect to x and y, which is a solution of the inequality

$$\omega\Big(x, y, \int_0^x \int_0^y g(u,v)\,du\,dv, \int_0^y g(x,v)\,dv, \int_0^x g(u,y)\,du, g(kx, ly)\Big) +$$
$$+h(x,y) \leqslant g(x,y),$$

where $h(x,y)$ is defined as in (7) and k, l are given constants, $0 \leqslant k \leqslant 1$, $0 \leqslant l \leqslant 1$;

2° in the class of functions satisfying the condition

$$0 \leqslant g(x,y) \leqslant w(x,y), \quad (x,y) \in \Delta,$$

the function $g(x,y) \equiv 0$, $(x,y) \in \Delta$, is the only measurable solution of the equation

$$g(x,y) = \omega\Big(x, y, \int_0^x \int_0^y g(u,v)\,du\,dv, \int_0^y g(x,v)\,dv, \int_0^x g(u,y)\,du, g(kx, ly)\Big).$$

Let us define the sequence $\{g_n(x,y)\}$, $(x,y) \in \Delta$, by the relations

$$(9) \quad \begin{cases} g_0(x,y) = \bar{g}(x,y), \\ g_{n+1}(x,y) = \omega\Big(x, y, \displaystyle\int_0^{\gamma_1(x,y)} \int_0^{\delta_1(x,y)} g_n(u,v)\,du\,dv, \int_0^{\delta_2(x,y)} g_n\big(\gamma_2(x,y), v\big)\,dv, \\ \qquad \displaystyle\int_0^{\gamma_3(x,y)} g_n\big(u, \delta_3(x,y)\big)\,du,\ g_n\big(\alpha(x,y), \beta(x,y)\big)\Big), \\ \qquad (x,y) \in \Delta,\ n = 0, 1, \ldots \end{cases}$$

LEMMA 1. *If assumptions 3° of H_1 and H_2 are satisfied, then*

$$(10) \quad 0 \leqslant g_{n+1}(x,y) \leqslant g_n(x,y) \leqslant \bar{g}(x,y), \quad (x,y) \in \Delta,\ n = 0, 1, \ldots,$$
$$g_n \xrightarrow{u} 0,$$

where the sign $\xrightarrow{u}$ denotes the uniform convergence.

Proof. From relations (7) and (9) we get

$$g_1(x,y) = \omega\Big(x, y, \int_0^{\gamma_1(x,y)}\int_0^{\delta_1(x,y)} g_0(u,v)\,du\,dv, \int_0^{\delta_2(x,y)} g_0\big(\gamma_2(x,y),v\big)dv,$$
$$\int_0^{\gamma_3(x,y)} g_0\big(u,\delta_3(x,y)\big)du, g_0\big(\alpha(x,y),\beta(x,y)\big)\Big)$$
$$\leqslant \omega\Big(x, y, \int_0^{\gamma_1(x,y)}\int_0^{\delta_1(x,y)} \bar{g}(u,v)\,du\,dv, \int_0^{\delta_2(x,y)} \bar{g}\big(\gamma_2(x,y),v\big)dv,$$
$$\int_0^{\gamma_3(x,y)} \bar{g}\big(u,\delta_3(x,y)\big)du, \bar{g}\big(\alpha(x,y),\beta(x,y)\big)\Big) + h(x,y)$$
$$\leqslant \bar{g}(x,y) = g_0(x,y) \quad \text{for } (x,y)\in\Delta.$$

Further, if we suppose that

$$g_n(x,y) \leqslant g_{n-1}(x,y) \leqslant \bar{g}(x,y), \quad (x,y)\in\Delta,$$

then

$$g_{n+1}(x,y) = \omega\Big(x, y, \int_0^{\gamma_1(x,y)}\int_0^{\delta_1(x,y)} g_n(u,v)\,du\,dv, \int_0^{\delta_2(x,y)} g_n\big(\gamma_2(x,y),v\big)dv,$$
$$\int_0^{\gamma_3(x,y)} g_n\big(u,\delta_3(x,y)\big)du, g_n\big(\alpha(x,y),\beta(x,y)\big)\Big)$$
$$\leqslant \omega\Big(x, y, \int_0^{\gamma_1(x,y)}\int_0^{\delta_1(x,y)} g_{n-1}(u,v)\,du\,dv, \int_0^{\delta_2(x,y)} g_{n-1}\big(\gamma_2(x,y),v\big)dv,$$
$$\int_0^{\gamma_3(x,y)} g_{n-1}\big(u,\delta_3(x,y)\big)du, g_{n-1}\big(\alpha(x,y),\beta(x,y)\big)\Big)$$
$$= g_n(x,y) \leqslant \bar{g}(x,y), \quad (x,y)\in\Delta.$$

Since the sequence of continuous functions g_n is non-increasing and bounded from below, it is convergent to a certain measurable function φ such that $0 \leqslant \varphi(x,y) \leqslant \bar{g}(x,y)$ for $(x,y)\in\Delta$. By the Lebesgue theorem and the continuity of ω it follows that the function φ satisfies equation (8).

Now, from assumption H_2, we have $\varphi(x,y) \equiv 0$, $(x,y)\in\Delta$.

The uniform convergence of the sequence $\{g_n\}$ in Δ follows from the Dini theorem. Thus the proof of Lemma 1 is complete.

Let us define the sequence $\{\tilde{g}_n(x,y)\}$, $(x,y)\in\Delta$, by the relations

$$(11)\quad \begin{cases} \tilde{g}_0(x,y) = w(x,y), \\ \tilde{g}_{n+1}(x,y) = \omega\Big(x, y, \int_0^x\int_0^y \tilde{g}_n(u,v)\,du\,dv, \int_0^y \tilde{g}_n(x,v)\,dv, \\ \qquad \int_0^x \tilde{g}_n(u,y)\,du, \tilde{g}_n(kx,ly)\Big), \quad (x,y)\in\Delta,\ n = 0,1,\ldots \end{cases}$$

We have then

LEMMA 2. *If assumption* H_3 *is satisfied, and*

1° $0 \leqslant \gamma_i(x, y) \leqslant x$, $0 \leqslant \delta_i(x, y) \leqslant y$, $i = 1, 2, 3$, $0 \leqslant a(x, y) \leqslant kx$, $0 \leqslant \beta(x, y) \leqslant ly$, $0 \leqslant k \leqslant 1$, $0 \leqslant l \leqslant 1$, $(x, y) \in \Delta$,

2° *the function* ω *is non-decreasing with respect to* x, y, z, p, q *and* r, *then*

(i) *the functions* $\tilde{g}_n$, $n = 0, 1, \ldots$, *are non-decreasing with respect to* x *and* y,

(ii) $0 \leqslant \tilde{g}_{n+1}(x, y) \leqslant \tilde{g}_n(x, y) \leqslant w(x, y)$ *for* $(x, y) \in \Delta$, $n = 0, 1, \ldots$, $\tilde{g} \xrightarrow{u} 0$ *in* Δ,

(iii) *the function* w *satisfies inequality* (7) *and if* $\bar{g}(x, y) \leqslant w(x, y)$, *then* $0 \leqslant g_n(x, y) \leqslant \tilde{g}_n(x, y)$, $(x, y) \in \Delta$, $n = 0, 1, \ldots$

2. The existence of a solution of equation (5). In order to prove the existence of a solution of equation (5) we shall show that the sequence $\{z_n\}$ defined by the relations

$$
(12) \quad \begin{cases} z_0(x, y) \equiv 0, \\ z_{n+1}(x, y) = f\Big(x, y, \int\limits_0^{\gamma_1(x,y)} \int\limits_0^{\delta_1(x,y)} z_n(u, v)\,du\,dv, \int\limits_0^{\delta_2(x,y)} z_n\big(\gamma_2(x, y), v\big)\,dv, \\ \qquad \int\limits_0^{\gamma_3(x,y)} z_n\big(u, \delta_3(x, y)\big)\,du, z_n\big(a(x, y), \beta(x, y)\big)\Big) \\ \qquad\qquad \text{for } (x, y) \in \Delta, n = 0, 1, \ldots \end{cases}
$$

is uniformly convergent to a solution of equation (5).

THEOREM 1. *If assumptions* H_1 *and* H_2 *are satisfied, then in the set* Δ *there exists a continuous solution* $\bar{z}$ *of equation* (5). *The sequence* $\{z_n\}$ *defined by* (12) *converges in* Δ *uniformly to* $\bar{z}$, *and the estimations*

$$
(13) \qquad |\bar{z}(x, y) - z_n(x, y)| \leqslant g_n(x, y), \quad (x, y) \in \Delta, \ n = 0, 1, \ldots,
$$

and

$$
(14) \qquad |\bar{z}(x, y)| \leqslant \bar{g}(x, y), \quad (x, y) \in \Delta,
$$

hold true.

A solution $\bar{z}$ *of* (5) *is unique in the class of functions satisfying* (14).

Proof. First, we shall prove that the sequence $\{z_n(x, y)\}$, $(x, y) \in \Delta$, fulfils the condition

$$
(15) \qquad |z_n(x, y)| \leqslant \bar{g}(x, y), \quad (x, y) \in \Delta, \ n = 0, 1, \ldots
$$

Evidently,

$$
|z_0(x, y)| \equiv 0 \leqslant \bar{g}(x, y), \quad (x, y) \in \Delta.
$$

Let us suppose that inequality (15) is true for $n \geqslant 0$. By the definition of $z_n(x, y)$, $(x, y) \in \Delta$, and, by condition 3° of H_1, we have

$$|z_{n+1}(x, y)| = \Big|f\Big(x, y, \int_0^{\gamma_1(x,y)} \int_0^{\delta_1(x,y)} z_n(u, v)\,du\,dv, \int_0^{\delta_2(x,y)} z_n\big(\gamma_2(x, y), v\big)dv,$$

$$\int_0^{\gamma_3(x,y)} z_n\big(u, \delta_3(x, y)\big)du, z_n\big(\alpha(x, y), \beta(x, y)\big)\Big) -$$

$$-f(x, y, 0, 0, 0, 0) + f(x, y, 0, 0, 0, 0)\Big|$$

$$\leqslant \omega\Big(x, y, \int_0^{\gamma_1(x,y)} \int_0^{\delta_1(x,y)} |z_n(u, v)|\,du\,dv, \int_0^{\delta_2(x,y)} \big|z_n\big(\gamma_2(x, y), v\big)\big|dv,$$

$$\int_0^{\gamma_3(x,y)} \big|z_n\big(u, \delta_3(x, y)\big)\big|du, \big|z_n\big(\alpha(x, y), \beta(x, y)\big)\big|\Big) + h(x, y)$$

$$\leqslant \omega\Big(x, y, \int_0^{\gamma_1(x,y)} \int_0^{\delta_1(x,y)} \bar{g}(u, v)\,du\,dv, \int_0^{\delta_2(x,y)} \bar{g}\big(\gamma_2(x, y), v\big)dv,$$

$$\int_0^{\gamma_3(x,y)} \bar{g}\big(u, \delta_3(x, y)\big)du, \bar{g}\big(\alpha(x, y), \beta(x, y)\big)\Big) + h(x, y)$$

$$\leqslant \bar{g}(x, y)$$

for $(x, y) \in \Delta$. Now (15) follows by induction.

Next, we prove, again by induction, that

$$(16) \qquad |z_{n+r}(x, y) - z_n(x, y)| \leqslant g_n(x, y), \qquad (x, y) \in \Delta,\ n, r = 0, 1, \ldots$$

By (15),

$$|z_r(x, y) - z_0(x, y)| = |z_r(x, y)| \leqslant \bar{g}(x, y) = g_0(x, y),$$
$$(x, y) \in \Delta,\ r = 0, 1, \ldots$$

If we suppose (16) to be true for $n, r \geqslant 0$, then

$$|z_{n+r+1}(x, y) - z_{n+1}(x, y)|$$

$$= \Big|f\Big(x, y, \int_0^{\gamma_1(x,y)} \int_0^{\delta_1(x,y)} z_{n+r}(u, v)\,du\,dv, \int_0^{\delta_2(x,y)} z_{n+r}\big(\gamma_2(x, y), v\big)dv,$$

$$\int_0^{\gamma_3(x,y)} z_{n+r}\big(u, \delta_3(x, y)\big)du, z_{n+r}\big(\alpha(x, y), \beta(x, y)\big)\Big) -$$

$$-f\Big(x, y, \int_0^{\gamma_1(x,y)} \int_0^{\delta_1(x,y)} z_n(u, v)\,du\,dv, \int_0^{\delta_2(x,y)} z_n\big(\gamma_2(x, y), v\big)dv,$$

$$\int_0^{\gamma_3(x,y)} z_n\big(u, \delta_3(x, y)\big)du, z_n\big(\alpha(x, y), \beta(x, y)\big)\Big)\Big|$$

$$\leqslant \omega\Big(x, y, \int\limits_0^{\gamma_1(x,y)} \int\limits_0^{\delta_1(x,y)} |z_{n+r}(u, v) - z_n(u, v)|\,du\,dv, \int\limits_0^{\delta_2(x,y)} \big|z_{n+r}\big(\gamma_2(x, y), v\big) -$$

$$- z_n\big(\gamma_2(x, y), v\big)\big|\,dv, \int\limits_0^{\gamma_3(x,y)} \big|z_{n+r}\big(u, \delta_3(x, y)\big) - z_n\big(u, \delta_3(x, y)\big)\big|\,du,$$

$$\big|z_{n+r}\big(a(x, y), \beta(x, y)\big) - z_n\big(a(x, y), \beta(x, y)\big)\big|\Big)$$

$$\leqslant \omega\Big(x, y, \int\limits_0^{\gamma_1(x,y)} \int\limits_0^{\delta_1(x,y)} g_n(u, v)\,du\,dv, \int\limits_0^{\delta_2(x,y)} g_n\big(\gamma_2(x, y), v\big)\,dv,$$

$$\int\limits_0^{\gamma_3(x,y)} g_n\big(u, \delta_3(x, y)\big)\,du, g_n\big(a(x, y), \beta(x, y)\big)\Big)$$

$$= g_{n+1}(x, y),$$

which completes the inductive step of the proof of (16).

By Lemma 1 we have $g_n \xrightarrow{u} 0$ in Δ, whence and from (16) we infer that $z_n \xrightarrow{u} \bar{z}$ in Δ. The continuity of $\bar{z}$ follows from the uniform convergence of the sequence $\{z_n\}$ and from the continuity of all functions z_n.

If $r \to \infty$, then (16) gives estimation (13). Estimation (14) is implied by (15). It is obvious that $\bar{z}$ is a solution of (5).

To prove that the solution $\bar{z}$ is unique in the considered class let us suppose that there exists another solution $\tilde{z}$ defined in Δ and such that $\bar{z}(x, y) \not\equiv z(x, y)$ and $|\tilde{z}(x, y)| \leqslant \bar{g}(x, y)$ for each $(x, y) \in \Delta$.

By induction we get

$$|\tilde{z}(x, y) - z_n(x, y)| \leqslant g_n(x, y), \quad (x, y) \in \Delta,\ n = 0, 1, \ldots,$$

whence it follows that $\bar{z}(x, y) \equiv \tilde{z}(x, y)$, $(x, y) \in \Delta$. This contradiction proves the uniqueness of $\bar{z}$, and so the proof of Theorem 1 is completed.

Now we can formulate the analogous theorem for the equation of Volterra's type.

THEOREM 2. *If assumptions* H_1, H_3, *and* 1°-2° *of Lemma* 2 *are fulfilled, then the assertion of Theorem* 1 *is true with* $w(x, y)$ *instead of* $\bar{g}(x, y)$, *and the estimations*

$$|\bar{z}(x, y) - z_n(x, y)| \leqslant \tilde{g}_n(x, y), \quad (x, y) \in \Delta,\ n = 0, 1, \ldots,$$

and

$$|\bar{z}(x, y)| \leqslant w(x, y), \quad (x, y) \in \Delta,$$

also hold true.

Proof. We prove that assumption H_2 is fulfilled. Since the function w is non-decreasing, it satisfies the same inequality as $\bar{g}$. Let g be a measurable solution of (8) in the class $0 \leqslant g(x, y) \leqslant w(x, y)$, $(x, y) \in \Delta$.

By induction we get

$$0 \leqslant g(x, y) \leqslant \tilde{g}_n(x, y), \quad (x, y) \in \Delta, \; n = 0, 1, \ldots,$$

and because $\tilde{g}_n(x, y) \to 0$ for $(x, y) \in \Delta$, then $g(x, y) \equiv 0$, $(x, y) \in \Delta$. Hence assumptions of Theorem 1 are fulfilled and $g_n(x, y) \leqslant \tilde{g}_n(x, y)$, $(x, y) \in \Delta$. Then the proof of Theorem 2 is complete.

3. Uniqueness theorem. Now we shall give conditions under which equation (5) has at most one solution (conditions will not guarantee the existence).

THEOREM 3. *If assumption* H_1 *is satisfied and the function* m, $m(x, y) \equiv 0$ *for* $(x, y) \in \Delta$, *is the only non-negative, finite and measurable solution of the inequality*

$$(17) \quad m(x, y) \leqslant \omega\Big(x, y, \int_0^{\gamma_1(x,y)} \int_0^{\delta_1(x,y)} m(u, v)\,du\,dv, \int_0^{\delta_2(x,y)} m\big(\gamma_2(x, y), v\big)dv, \int_0^{\gamma_3(x,y)} m\big(u, \delta_3(x, y)\big)du, m\big(\alpha(x, y), \beta(x, y)\big)\Big), \quad (x, y) \in \Delta,$$

then equation (5) *has at most one solution in the set* Δ.

Proof. Let us suppose that there exist two solutions $\tilde{z}$ and $\tilde{\tilde{z}}$ of equation (5) in Δ, $\tilde{z}(x, y) \not\equiv \tilde{\tilde{z}}(x, y)$.

Now, from condition 3° of H_1, we have

$$\begin{aligned}
|\tilde{z}(x, y) - \tilde{\tilde{z}}(x, y)| &= \Big| f\Big(x, y, \int_0^{\gamma_1(x,y)} \int_0^{\delta_1(x,y)} \tilde{z}(u, v)\,du\,dv, \int_0^{\delta_2(x,y)} \tilde{z}\big(\gamma_2(x, y), v\big)dv, \\
&\qquad \int_0^{\gamma_3(x,y)} \tilde{z}\big(u, \delta_3(x, y)\big)du, \tilde{z}\big(\alpha(x, y), \beta(x, y)\big)\Big) - \\
&\quad - f\Big(x, y, \int_0^{\gamma_1(x,y)} \int_0^{\delta_1(x,y)} \tilde{\tilde{z}}(u, v)\,du\,dv, \int_0^{\delta_2(x,y)} \tilde{\tilde{z}}\big(\gamma_2(x, y), v\big)dv, \\
&\qquad \int_0^{\gamma_3(x,y)} \tilde{\tilde{z}}\big(u, \delta_3(x, y)\big)du, \tilde{\tilde{z}}\big(\alpha(x, y), \beta(x, y)\big)\Big)\Big| \\
&\leqslant \omega\Big(x, y, \int_0^{\gamma_1(x,y)} \int_0^{\delta_1(x,y)} |\tilde{z}(u, v) - \tilde{\tilde{z}}(u, v)|\,du\,dv, \\
&\qquad \int_0^{\delta_2(x,y)} \big|\tilde{z}\big(\gamma_2(x, y), v\big) - \tilde{\tilde{z}}\big(\gamma_2(x, y), v\big)\big|dv, \int_0^{\gamma_3(x,y)} \big|\tilde{z}\big(u, \delta_3(x, y)\big) - \\
&\qquad - \tilde{\tilde{z}}\big(u, \delta_3(x, y)\big)\big|du, \big|\tilde{z}\big(\alpha(x, y), \beta(x, y)\big) - \\
&\qquad - \tilde{\tilde{z}}\big(\alpha(x, y), \beta(x, y)\big)\big|\Big), \quad (x, y) \in \Delta.
\end{aligned}$$

Putting

$$m(x,y) = |\tilde{z}(x,y)-\tilde{\tilde{z}}(x,y)|, \quad (x,y)\in\Delta,$$

we infer from (17) that $m(x,y)\equiv 0$ for $(x,y)\in\Delta$, i.e., $\tilde{z}(x,y)\equiv\tilde{\tilde{z}}(x,y)$, $(x,y)\in\Delta$. This contradiction proves Theorem 3.

Remark 3. If assumption H_2 is satisfied, then the function m, $m(x,y)\equiv 0$ for $(x,y)\in\Delta$, is the only measurable solution of (17) in the class $0\leqslant m(x,y)\leqslant \bar{g}(x,y)$, $(x,y)\in\Delta$.

Indeed, we can prove by induction that

$$0\leqslant m(x,y)\leqslant g_n(x,y), \quad (x,y)\in\Delta,\ n=0,1,\ldots,$$

and if $n\to\infty$, then, in view of Lemma 1, we have $m(x,y)\equiv 0$ for $(x,y)\in\Delta$.

Remark 4. If assumptions H_1 and 1°-2° of Lemma 2 are satisfied, and the function g, $g(x,y)\equiv 0$ for $(x,y)\in\Delta$, is the only non-negative, non-decreasing, finite and measurable solution of the inequality

$$(18)\qquad g(x,y)\leqslant\omega\Big(x,y,\int_0^x\int_0^y g(u,v)\,du\,dv,\int_0^y g(x,v)\,dv,$$
$$\int_0^x g(u,y)\,du,\ g(kx,ly)\Big),\quad (x,y)\in\Delta,$$

then equation (5) has at most one solution.

4. Continuous dependence of solutions on the right-hand side of equation (5). Let us consider the second equation

$$(19)\qquad p(x,y)=P\Big(x,y,\int_0^{\bar{\gamma}_1(x,y)}\int_0^{\bar{\delta}_1(x,y)} p(u,v)\,du\,dv,\int_0^{\bar{\delta}_2(x,y)} p\big(\bar{\gamma}_2(x,y),v\big)\,dv,$$
$$\int_0^{\bar{\gamma}_3(x,y)} p\big(u,\bar{\delta}_3(x,y)\big)\,du,\ p\big(\bar{\alpha}(x,y),\bar{\beta}(x,y)\big)\Big),$$

where functions $P,\bar{\gamma}_i,\bar{\delta}_i,\bar{\alpha},\bar{\beta}$ have the same properties as $f,\gamma_i,\delta_i,\alpha,\beta$ in assumption H_1, $i=1,2,3$.

THEOREM 4. *If assumption* H_1 *is satisfied, and*

1° $\bar{z}$ *and* $\bar{p}$ *are solutions of equations* (5) *and* (19), *respectively,*

2° *the sequence* $\{u_n(x,y)\}$, $(x,y)\in\Delta$, *defined by the relations*

$$u_0(x,y)\geqslant|\bar{z}(x,y)|+|\bar{p}(x,y)|,\quad (x,y)\in\Delta,$$

$$u_{n+1}(x,y)=\omega\Big(x,y,\int_0^{\gamma_1(x,y)}\int_0^{\delta_1(x,y)} u_n(u,v)\,du\,dv,\int_0^{\delta_2(x,y)} u_n\big(\gamma_2(x,y),v\big)\,dv,$$
$$\int_0^{\gamma_3(x,y)} u_n\big(u,\delta_3(x,y)\big)\,du,\ u_n\big(\alpha(x,y),\beta(x,y)\big)\Big)+\bar{h}(x,y),\quad n=0,1,\ldots,$$

$$\bar{h}(x,y) \overset{\mathrm{df}}{=} \Big|f\Big(x,y,\int_0^{\gamma_1(x,y)}\int_0^{\delta_1(x,y)} \bar{p}(u,v)\,du\,dv, \int_0^{\delta_2(x,y)} \bar{p}\big(\gamma_2(x,y),v\big)\,dv,$$
$$\int_0^{\gamma_3(x,y)} \bar{p}\big(u,\delta_3(x,y)\big)\,du, \bar{p}\big(\alpha(x,y),\beta(x,y)\big)\Big)-\bar{p}(x,y)\Big|,$$

has the limit $\bar{u}(x,y)$ *for* $(x,y)\in\Delta$,

then

$$|\bar{z}(x,y)-\bar{p}(x,y)| \leqslant \bar{u}(x,y), \qquad (x,y)\in\Delta. \tag{20}$$

Proof. Let

$$u(x,y) = |\bar{z}(x,y)-\bar{p}(x,y)|, \qquad (x,y)\in\Delta.$$

Thus, for $(x,y)\in\Delta$, we have

$$u(x,y) = \Big|f\Big(x,y,\int_0^{\gamma_1(x,y)}\int_0^{\delta_1(x,y)} \bar{z}(u,v)\,du\,dv, \int_0^{\delta_2(x,y)} \bar{z}\big(\gamma_2(x,y),v\big)\,dv,$$
$$\int_0^{\gamma_3(x,y)} \bar{z}\big(u,\delta_3(x,y)\big)\,du, \bar{z}\big(\alpha(x,y),\beta(x,y)\big)\Big)-\bar{p}(x,y)\Big|$$
$$\leqslant \Big|f\Big(x,y,\int_0^{\gamma_1(x,y)}\int_0^{\delta_1(x,y)} \bar{z}(u,v)\,du\,dv, \int_0^{\delta_2(x,y)} \bar{z}\big(\gamma_2(x,y),v\big)\,dv,$$
$$\int_0^{\gamma_3(x,y)} \bar{z}\big(u,\delta_3(x,y)\big)\,du, \bar{z}\big(\alpha(x,y),\beta(x,y)\big)\Big)-$$
$$-f\Big(x,y,\int_0^{\gamma_1(x,y)}\int_0^{\delta_1(x,y)} \bar{p}(u,v)\,du\,dv, \int_0^{\delta_2(x,y)} \bar{p}\big(\gamma_2(x,y),v\big)\,dv,$$
$$\int_0^{\gamma_3(x,y)} \bar{p}\big(u,\delta_3(x,y)\big)\,du, \bar{p}\big(\alpha(x,y),\beta(x,y)\big)\Big)\Big|+$$
$$+\Big|f\Big(x,y,\int_0^{\gamma_1(x,y)}\int_0^{\delta_1(x,y)} \bar{p}(u,v)\,du\,dv, \int_0^{\delta_2(x,y)} \bar{p}\big(\gamma_2(x,y),v\big)\,dv,$$
$$\int_0^{\gamma_3(x,y)} \bar{p}\big(u,\delta_3(x,y)\big)\,du, \bar{p}\big(\alpha(x,y),\beta(x,y)\big)\Big)-\bar{p}(x,y)\Big|$$
$$\leqslant \omega\Big(x,y,\int_0^{\gamma_1(x,y)}\int_0^{\delta_1(x,y)} |\bar{z}(u,v)-\bar{p}(u,v)|\,du\,dv,$$
$$\int_0^{\delta_2(x,y)} \big|\bar{z}\big(\gamma_2(x,y),v\big)-\bar{p}\big(\gamma_2(x,y),v\big)\big|\,dv, \int_0^{\gamma_3(x,y)} \big|\bar{z}\big(u,\delta_3(x,y)\big)-\bar{p}\big(u,\delta_3(x,y)\big)\big|\,du,$$
$$\big|\bar{z}\big(\alpha(x,y),\beta(x,y)\big)-\bar{p}\big(\alpha(x,y),\beta(x,y)\big)\big|\Big)+\bar{h}(x,y).$$

Since

$$u(x, y) \leqslant |\bar{z}(x, y)| + |\bar{p}(x, y)| \leqslant u_0(x, y), \qquad (x, y) \in \Delta,$$

we get by induction

$$u(x, y) \leqslant u_n(x, y), \qquad (x, y) \in \Delta, \ n = 0, 1, \ldots$$

Inequality (20) is implied by the last one for $n \to \infty$.

Remark 5. If functions u_n, $n = 0, 1, \ldots$, are finite and measurable, and if there exists a Lebesgue-integrable function T: $\Delta \to R^1_+$ such that

$$|u_n(x, y)| \leqslant T(x, y), \qquad (x, y) \in \Delta, \ n = 0, 1, \ldots,$$

then the limit function $\bar{u}$ (see 2° of Theorem 4) is a finite and measurable solution of the equation

$$u(x, y) = \omega\Big(x, y, \int\limits_0^{\gamma_1(x,y)} \int\limits_0^{\delta_1(x,y)} u(u, v)\,du\,dv, \int\limits_0^{\delta_2(x,y)} u\big(\gamma_2(x, y), v\big)\,dv,$$
$$\int\limits_0^{\gamma_3(x,y)} u\big(u, \delta_3(x, y)\big)\,du, u\big(a(x, y), \beta(x, y)\big)\Big) + \bar{h}(x, y), \qquad (x, y) \in \Delta.$$

Remark 6. It follows from the proof of Theorem 4 that this theorem is true if in the set Δ there exists a non-negative and continuous function k_0 satisfying the inequality

$$\omega\Big(x, y, \int\limits_0^{\gamma_1(x,y)} \int\limits_0^{\delta_1(x,y)} k_0(u, v)\,du\,dv, \int\limits_0^{\delta_2(x,y)} k_0\big(\gamma_2(x, y), v\big)\,dv,$$
$$\int\limits_0^{\gamma_3(x,y)} k_0\big(u, \delta_3(x, y)\big)\,du, k_0\big(a(x, y), \beta(x, y)\big)\Big) +$$
$$+ \max\big[\bar{h}(x, y), u_0(x, y)\big] \leqslant k_0(x, y), \qquad (x, y) \in \Delta.$$

Now, in the class of measurable functions satisfying the condition $0 \leqslant g(x, y) \leqslant k_0(x, y)$, $(x, y) \in \Delta$, there exists a function $\bar{k}$ being a solution of the equation

$$\omega\Big(x, y, \int\limits_0^{\gamma_1(x,y)} \int\limits_0^{\delta_1(x,y)} g(u, v)\,du\,dv, \int\limits_0^{\delta_2(x,y)} g\big(\gamma_2(x, y), v\big)\,dv,$$
$$\int\limits_0^{\gamma_3(x,y)} g\big(u, \delta_3(x, y)\big)\,du, g\big(a(x, y), \beta(x, y)\big)\Big) + \bar{h}(x, y) = g(x, y), \qquad (x, y) \in \Delta.$$

Put

$$k_{n+1}(x,y) = \omega\Big(x, y, \int_0^{\gamma_1(x,y)}\int_0^{\delta_1(x,y)} k_n(u,v)\,du\,dv,$$

$$\int_0^{\delta_2(x,y)} k_n\big(\gamma_2(x,y),v\big)dv, \int_0^{\gamma_3(x,y)} k_n\big(u,\delta_3(x,y)\big)du, k_n\big(\alpha(x,y),\beta(x,y)\big)\Big)+$$

$$+\bar{h}(x,y), \quad (x,y)\in \Delta,\ n = 0,1,\ldots$$

We see that

$$u_n(x,y)\leqslant k_n(x,y), \quad k_{n+1}(x,y)\leqslant k_n(x,y), \quad (x,y)\in\Delta,\ n = 0,1,\ldots,$$

whence $u(x,y)\leqslant k_n(x,y)$, $(x,y)\in\Delta$, $n = 0,1,\ldots$ ($u(x,y)$ is defined as in the proof of Theorem 4). From the last inequality we get $u_n(x,y) \to \bar{u}(x,y)$, and $u(x,y)\leqslant \bar{u}(x,y)\leqslant \bar{k}(x,y)$ for $(x,y)\in\Delta$.

THEOREM 5. *If assumptions of Theorem 4 (except for 2°) and 1°-2° of Lemma 2 are satisfied, and the sequence $\{\tilde{u}_n(x,y)\}$ defined by the relations*

$$\tilde{u}_0(x,y) = \sup_{0\leqslant\xi\leqslant x}\ \sup_{0\leqslant\eta\leqslant y}\{|\bar{z}(\xi,\eta)|+|\bar{p}(\xi,\eta)|\}, \quad (x,y)\in\Delta,$$

$$\tilde{u}_{n+1}(x,y) = \omega\Big(x,y,\int_0^x\int_0^y \tilde{u}_n(u,v)\,du\,dv, \int_0^y \tilde{u}_n(x,v)\,dv,$$

$$\int_0^x \tilde{u}_n(u,y)\,du, \tilde{u}_n(kx,ly)\Big)+\sup_{0\leqslant\xi\leqslant x}\ \sup_{0\leqslant\eta\leqslant y}\bar{h}(\xi,\eta)$$

for $(x,y)\in\Delta$, $n = 0,1,\ldots$, has the limit $\tilde{u}(x,y)$, $(x,y)\in\Delta$, then

$$|\bar{z}(x,y)-\bar{p}(x,y)|\leqslant \tilde{u}(x,y), \quad (x,y)\in\Delta. \tag{21}$$

Proof. It is obvious that functions $\tilde{u}_n$ are non-decreasing with respect to x and y for $(x,y)\in\Delta$, $n = 0,1\ldots$ Further, we get by induction $u_n(x,y) \leqslant \tilde{u}_n(x,y)$, $(x,y)\in\Delta$, $n = 0,1,\ldots,$ where the sequence $\{u_n(x,y)\}$ is defined as in 2° of Theorem 4. Hence $u(x,y)\leqslant\tilde{u}_n(x,y)$, $(x,y)\in\Delta$, $n = 0,1\ldots$ ($u(x,y)$ is defined as in the proof of Theorem 4), and if $n\to\infty$, we have (21).

5. The case of a function ω linear in r. At first, we assume

$$\omega(x,y,z,p,q,r) = \lambda(x,y)\cdot r, \quad \lambda(x,y)\geqslant 0,\ (x,y)\in\Delta.$$

In this case equation (1) is purely functional, but its discussion is needed in the sequel.

Let

$$(22)\quad \begin{cases} a_0(x, y) = x, & a_{n+1}(x, y) = a\big(a_n(x, y), \beta_n(x, y)\big), \\ \beta_0(x, y) = y, & \beta_{n+1}(x, y) = \beta\big(a_n(x, y), \beta_n(x, y)\big), \\ \lambda_0(x, y) = 1, & \lambda_{n+1}(x, y) = \prod_{i=0}^{n} \lambda\big(a_i(x, y), \beta_i(x, y)\big), \\ & \qquad (x, y) \in \Delta,\ n = 1, 2, \ldots, \end{cases}$$

where $a(x, y)$ and $\beta(x, y)$, $(x, y) \in \Delta$, satisfy assumption H_1.

It is obvious that $a_n(x, y) \in [0, a]$ and $\beta_n(x, y) \in [0, b]$ for $(x, y) \in \Delta$, $n = 0, 1, \ldots$

Now we formulate lemmas by which assumption H_2 is fulfilled in this special case.

LEMMA 3. *For any function h: $\Delta \to R^1_+$ the condition*

$$(23)\quad \sum_{n=0}^{\infty} \lambda_n(x, y) h\big(a_n(x, y),\ \beta_n(x, y)\big) < \infty, \qquad (x, y) \in \Delta,$$

is necessary and sufficient for the equation

$$(24)\quad g(x, y) = \lambda(x, y) g\big(a(x, y),\ \beta(x, y)\big) + h(x, y), \qquad (x, y) \in \Delta,$$

to have a non-negative solution $\bar{g}$ defined in Δ.

If condition (23) *is fulfilled, then the function $\bar{g}$,*

$$(25)\quad \bar{g}(x, y) = \sum_{n=0}^{\infty} \lambda_n(x, y) h\big(a_n(x, y),\ \beta_n(x, y)\big), \qquad (x, y) \in \Delta,$$

is a solution of equation (24), *and*

$$(26)\quad \lim_{n\to\infty} \lambda_n(x, y) \bar{g}\big(a_n(x, y),\ \beta_n(x, y)\big) = 0, \qquad (x, y) \in \Delta.$$

There is no other solution of equation (24) *in the class of functions g satisfying the condition* $0 \leqslant g(x, y) \leqslant \bar{g}(x, y)$.

Proof. Necessity. Let g be a non-negative solution of equation (24). We prove that, for $(x, y) \in \Delta$ and $n = 0, 1, \ldots,$

$$(27)\quad g(x, y) = \sum_{i=0}^{n} \lambda_i(x, y) h\big(a_i(x, y),\ \beta_i(x, y)\big) + \\ + \lambda_{n+1}(x, y) g\big(a_{n+1}(x, y),\ \beta_{n+1}(x, y)\big).$$

Evidently, formula (27) is true for $n = 0$. Suppose it to be true for $n \geqslant 0$. By the definitions of sequences $\{a_n(x, y)\}$, $\{\beta_n(x, y)\}$ and $\{\lambda_n(x, y)\}$, we have

$$
\begin{aligned}
g(x, y) &= \lambda(x, y) g\big(a(x, y),\ \beta(x, y)\big) + h(x, y) \\
&= \lambda(x, y) \sum_{i=0}^{n} \lambda_i\big(a(x, y), \beta(x, y)\big) h\Big(a_i\big(a(x, y), \beta(x, y)\big), \\
&\qquad \beta_i\big(a(x, y), \beta(x, y)\big)\Big) + \\
&\quad + \lambda(x, y)\lambda_{n+1}\big(a(x, y), \beta(x, y)\big) g\Big(a_{n+1}\big(a(x, y), \beta(x, y)\big), \\
&\qquad \beta_{n+1}\big(a(x, y), \beta(x, y)\big)\Big) + h(x, y) \\
&= \sum_{i=1}^{n+1} \lambda_i(x, y) h\big(a_i(x, y), \beta_i(x, y)\big) + \lambda_0(x, y) h\big(a_0(x, y), \\
&\qquad \beta_0(x, y)\big) + \lambda_{n+2}(x, y) g\big(a_{n+2}(x, y), \beta_{n+2}(x, y)\big) \\
&= \sum_{i=0}^{n+1} \lambda_i(x, y) h\big(a_i(x, y), \beta_i(x, y)\big) + \lambda_{n+2}(x, y) g\big(a_{n+2}(x, y), \\
&\qquad \beta_{n+2}(x, y)\big), \qquad (x, y) \in \Delta.
\end{aligned}
$$

Now we obtain (27) by induction.

Since g is a non-negative solution of equation (24), we infer from condition (27) that

$$g(x, y) \geqslant \sum_{i=0}^{n} \lambda_i(x, y) h\big(a_i(x, y), \beta_i(x, y)\big), \qquad (x, y) \in \Delta,$$

whence (23) follows for $n \to \infty$.

Sufficiency. If condition (23) is satisfied, then the function $\bar{g}$ defined by (25) is a solution of (24). It follows from the condition $\lambda(x, y) \geqslant 0$ for $(x, y) \in \Delta$ and from the definition of the sequence $\{\lambda_n(x, y)\}$ that $\bar{g}$ is non-negative. Since each solution of (24) fulfils condition (27) and $\bar{g}$ is a solution of (24), then (27) yields (26) for $n \to \infty$.

It remains to show that no other function $\tilde{g}$, $0 \leqslant \tilde{g}(x, y) \leqslant \bar{g}(x, y)$, $(x, y) \in \Delta$, is a solution of (24). Assume a contrario that a function $\tilde{g}$ satisfying the last inequality and distinct from $\bar{g}$ is also a solution of (24). Since condition (27) is fulfilled for the solution $\tilde{g}$, we get

$$0 \leqslant \bar{g}(x, y) - \tilde{g}(x, y) \leqslant \lim_{n \to \infty} \lambda_n(x, y) \bar{g}\big(a_n(x, y), \beta_n(x, y)\big), \qquad (x, y) \in \Delta.$$

Now according to (26) we have $\bar{g}(x, y) \equiv \tilde{g}(x, y)$. This contradiction proves the uniqueness of the solution $\bar{g}$ of equation (24) in the class of functions satisfying the condition $0 \leqslant g(x, y) \leqslant \bar{g}(x, y)$, $(x, y) \in \Delta$.

LEMMA 4. *If*

1° $0 \leqslant \varphi_1(x, y) \leqslant \varphi_2(x, y)$, $(x, y) \in \Delta$, 2° $\sum_{n=0}^{\infty}\lambda_n(x,y)\varphi_2\big(a_n(x,y),\beta_n(x,y)\big) < \infty,\ (x,y)\in\Delta,$

then the functions

$$\bar{v}_i(x,y) = \sum_{n=0}^{\infty}\lambda_n(x,y)\varphi_i\big(a_n(x,y),\beta_n(x,y)\big), \quad (x,y)\in\Delta,\ i=1,2,$$

are non-negative solutions of the equations

$$(28)\quad v(x,y) = \lambda(x,y)v\big(a(x,y),\beta(x,y)\big)+\varphi_i(x,y), \quad (x,y)\in\Delta, i=1,2,$$

respectively, and

$$(29)\quad \lim_{n\to\infty}\lambda_n(x,y)\bar{v}_i\big(a_n(x,y),\beta_n(x,y)\big) = 0, \quad (x,y)\in\Delta,\ i=1,2.$$

Moreover, functions $\bar{v}_i$, $i=1,2$, *are unique solutions of* (28) *in the class of functions satisfying* $0\leqslant v(x,y)\leqslant \bar{v}_2(x,y)$, $(x,y)\in\Delta$.

Proof. It follows from Lemma 3 that the function $\bar{v}_1$ is the unique solution of (28) in the class $0\leqslant v(x,y)\leqslant\bar{v}_1(x,y)$, $(x,y)\in\Delta$, and the function $\bar{v}_2$ is the unique solution of (28) in the class $0\leqslant v(x,y)\leqslant\bar{v}_2(x,y)$, $(x,y)\in\Delta$, and that (29) is true. It remains to prove that function $\bar{v}_1$ is the unique solution of (28) in the class $\bar{v}_1(x,y)\leqslant v(x,y)\leqslant\bar{v}_2(x,y)$, $(x,y)\in\Delta$. Assume that there exists another solution $\varkappa$ of (28) in this class, $\varkappa(x,y)\not\equiv\bar{v}_1(x,y)$ for $(x,y)\in\Delta$. Since any solution r_i of (28), $i=1,2$, satisfies the conditions

$$(30)\quad r_i(x,y) = \sum_{n=0}^{m}\lambda_n(x,y)\varphi_i\big(a_n(x,y),\beta_n(x,y)\big)+ \\ +\lambda_{m+1}(x,y)r_i\big(a_{m+1}(x,y),\beta_{m+1}(x,y)\big), \quad m=0,1,\ldots,$$

then, for $(x,y)\in\Delta$, we have

$$\begin{aligned}0\leqslant\varkappa(x,y)-\bar{v}_1(x,y) &= \lambda_{m+1}(x,y)\varkappa\big(a_{m+1}(x,y),\beta_{m+1}(x,y)\big)- \\ &\quad -\lambda_{m+1}(x,y)\bar{v}_1\big(a_{m+1}(x,y),\beta_{m+1}(x,y)\big) \\ &\leqslant \lambda_{m+1}(x,y)\bar{v}_2\big(a_{m+1}(x,y),\beta_{m+1}(x,y)\big).\end{aligned}$$

Now, if $m\to\infty$, we have $\varkappa(x,y)\equiv\bar{v}_1(x,y)$, $(x,y)\in\Delta$. This contradiction proves the uniqueness of the solution $\bar{v}_1(x,y)$, $(x,y)\in\Delta$, of (28) in the class of functions $0\leqslant v(x,y)\leqslant\bar{v}_2(x,y)$, $(x,y)\in\Delta$. These considerations and Theorem 1 imply

THEOREM 6. *If assumption* H_1 *is satisfied, and*

1° $\omega(x,y,z,p,q,r) = \lambda(x,y)r$, $\lambda(x,y)\geqslant 0$, $(x,y)\in\Delta$,

2° $\sum_{n=0}^{\infty} \lambda_n(x, y) h\big(a_n(x, y), \beta_n(x, y)\big) < \infty$, $(x, y) \in \Delta$, *where*

$$h(x, y) = \sup_{0 \leqslant \xi \leqslant x} \sup_{0 \leqslant \eta \leqslant y} |f(\xi, \eta, 0, 0, 0, 0)|, \qquad (x, y) \in \Delta,$$

then there exists in Δ *a solution* $\bar{z}$ *of equation* (5) *with the properties*

$$|\bar{z}(x, y)| \leqslant \bar{g}(x, y), \qquad (x, y) \in \Delta,$$

$$|\bar{z}(x, y) - z_n(x, y)| \leqslant g_n(x, y), \qquad (x, y) \in \Delta, \ n = 0, 1, \ldots,$$

where

$$g_0(x, y) = \bar{g}(x, y) = \sum_{i=0}^{\infty} \lambda_i(x, y) h\big(a_i(x, y), \beta_i(x, y)\big), \qquad (x, y) \in \Delta,$$

$$g_{n+1}(x, y) = \sum_{i=n}^{\infty} \lambda_i(x, y) h\big(a_i(x, y), \beta_i(x, y)\big), \qquad (x, y) \in \Delta, \ n = 0, 1, \ldots$$

The solution $\bar{z}$ *is unique in the class of functions satisfying the inequality* $|z(x, y)| \leqslant \bar{g}(x, y)$, $(x, y) \in \Delta$.

Theorem 4 implies the following

THEOREM 7. *If assumption* H_1 *is satisfied, and*

1° $\omega(x, y, z, p, q, r) = \lambda(x, y) r$, $(x, y) \in \Delta$,

2° *functions* $\bar{z}$ *and* $\bar{p}$ *are solutions of equations* (5) *and* (19), *respectively*,

3° $\sum_{n=0}^{\infty} \lambda_n(x, y) c\big(a_n(x, y), \beta_n(x, y)\big) < \infty$, $(x, y) \in \Delta$, *where*

$$c(x, y) \geqslant \max\{|\bar{z}(x, y)| + |\bar{p}(x, y)|, \bar{h}(x, y)\}, \qquad (x, y) \in \Delta,$$

and $\bar{h}(x, y)$ *is defined by condition* 2° *of Theorem* 4,

then

$$|\bar{z}(x, y) - \bar{p}(x, y)| \leqslant \sum_{n=0}^{\infty} \lambda_n(x, y) \bar{h}\big(a_n(x, y), \beta_n(x, y)\big), \qquad (x, y) \in \Delta.$$

6. Discussion of equation (5) being of Volterra's type for linear ω. Now we are going to consider the case

$$\omega(x, y, z, p, q, r) = Kz + Mp + Nq + \lambda r, \qquad (x, y) \in \Delta, \tag{31}$$

where K, M, N and λ are non-negative constants.

In this section we assume that the functions γ_i, δ_i, a, β, $i = 1, 2, 3$, satisfy the conditions

$$\begin{gathered} 0 \leqslant \gamma_i(x, y) \leqslant x, \quad 0 \leqslant \delta_i(x, y) \leqslant y, \quad i = 1, 2, 3, \\ 0 \leqslant a(x, y) \leqslant kx, \quad 0 \leqslant \beta(x, y) \leqslant ly, \\ 0 \leqslant k \leqslant 1, \ 0 \leqslant l \leqslant 1, \ (x, y) \in \Delta. \end{gathered} \tag{32}$$

Now, the sequences $\{\alpha_n(x, y)\}$, $\{\beta_n(x, y)\}$ and $\{\lambda_n(x, y)\}$, $(x, y) \in \Delta$, defined by (22), satisfy the relations

$$\alpha_n(x, y) \leqslant k^n x, \quad \beta_n(x, y) \leqslant l^n y, \quad \lambda_n(x, y) = \lambda^n, \quad (x, y) \in \Delta, \; n = 0, 1, \ldots \tag{33}$$

We have

LEMMA 5 [1]. *If* $t \in [0, +\infty)$ *and* $a \in [0, 1]$, *then*

$$e^{t(a-1)} \leqslant a(1-e^{-t}) + e^{-t}. \tag{34}$$

LEMMA 6. *If*

1° *the function*

$$H(x, y) \overset{\mathrm{df}}{=} \sum_{n=0}^{\infty} \lambda^n h(k^n x, l^n y) < \infty,$$

is continuous for $(x, y) \in \Delta$,

2° $0 \leqslant \lambda k < 1$, $0 \leqslant \lambda l < 1$,

3° $0 \leqslant k \leqslant 1$, $0 \leqslant l \leqslant 1$,

4° *the function* h *is continuous, non-negative and non-decreasing in the set* Δ,

then

(a) *there exists a unique solution* g^* *of the equation*

$$g(x, y) = K \sum_{n=0}^{\infty} \lambda^n \int_0^{k^n x} \int_0^{l^n y} g(u, v)\, du dv + M \sum_{n=0}^{\infty} \lambda^n \int_0^{l^n y} g(k^n x, v)\, dv + N \sum_{n=0}^{\infty} \lambda^n \int_0^{k^n x} g(u, l^n y)\, du + \sum_{n=0}^{\infty} \lambda^n h(k^n x, l^n y), \quad (x, y) \in \Delta, \tag{35}$$

and this solution is continuous, non-negative and non-decreasing in the set Δ,

(b) *in the class of measurable functions satisfying the condition* $0 \leqslant g(x, y) \leqslant g^*(x, y)$, $(x, y) \in \Delta$, *the function* g^* *is the unique, continuous, non-negative and non-decreasing solution of the equation*

$$g(x, y) = \lambda g(kx, ly) + K \int_0^x \int_0^y g(u, v)\, du dv + M \int_0^y g(x, y)\, dv + N \int_0^x g(u, y)\, du + h(x, y), \quad (x, y) \in \Delta, \tag{36}$$

(c) *in the class of measurable functions satisfying the condition* $0 \leqslant g(x, y) \leqslant g^*(x, y)$, $(x, y) \in \Delta$, *the function* g, $g(x, y) \equiv 0$ *for* $(x, y) \in \Delta$, *is the unique solution of the inequality*

$$(37)\quad g(x,y)\leqslant \lambda g(kx,ly)+K\int_0^x\int_0^y g(u,v)\,du\,dv+M\int_0^y g(x,v)\,dv+ \\ +N\int_0^x g(u,y)\,du,\qquad (x,y)\in \Delta.$$

Proof. Let A be the operator defined by the right-hand side of equation (35), and

$$\|g\| = \max_{\substack{0\leqslant x\leqslant a\\ 0\leqslant y\leqslant b}} e^{-L(x+y)}|g(x,y)| \qquad \text{for } g\in C_\Delta,$$

where

$$L>\frac{1}{2}\left\{\frac{M}{1-\lambda l}+\frac{N}{1-\lambda k}+\left[\left(\frac{M}{1-\lambda l}+\frac{N}{1-\lambda k}\right)^2+\frac{4K}{1-\lambda kl}\right]^{1/2}\right\}$$

and C_Δ denotes the class of continuous functions in Δ.

Now, in view of Lemma 5, we have, for $g, z\in C_\Delta$,

$$\|Ag-Az\|$$

$$\leqslant K\max_{\substack{0\leqslant x\leqslant a\\ 0\leqslant y\leqslant b}} e^{-L(x+y)}\left|\sum_{n=0}^{\infty}\lambda^n\int_0^{k^nx}\int_0^{l^ny}[g(u,v)-z(u,v)]e^{-L(u+v)}e^{L(u+v)}\,du\,dv\right|+$$

$$+M\max_{\substack{0\leqslant x\leqslant a\\ 0\leqslant y\leqslant b}} e^{-L(x+y)}\left|\sum_{n=0}^{\infty}\lambda^n\int_0^{l^ny}[g(k^nx,v)-z(k^nx,v)]e^{-L(k^nx+v)}e^{L(k^nx+v)}\,dv\right|+$$

$$+N\max_{\substack{0\leqslant x\leqslant a\\ 0\leqslant y\leqslant b}} e^{-L(x+y)}\left|\sum_{n=0}^{\infty}\lambda^n\int_0^{k^nx}[g(u,l^ny)-z(u,l^ny)]e^{-L(u+l^ny)}e^{L(u+l^ny)}\,du\right|$$

$$\leqslant\frac{K}{L^2}\|g-z\|\sum_{n=0}^{\infty}\lambda^n\max_{\substack{0\leqslant x\leqslant a\\ 0\leqslant y\leqslant b}}[e^{Lx(k^n-1)}-e^{-Lx}][e^{Ly(l^n-1)}-e^{-Ly}]+$$

$$+\frac{M}{L}\|g-z\|\sum_{n=0}^{\infty}\lambda^n\max_{\substack{0\leqslant x\leqslant a\\ 0\leqslant y\leqslant b}}e^{Lx(k^n-1)}[e^{Ly(k^n-1)}-e^{-Ly}]+$$

$$+\frac{N}{L}\|g-z\|\sum_{n=0}^{\infty}\lambda^n\max_{\substack{0\leqslant x\leqslant a\\ 0\leqslant y\leqslant b}}e^{Ly(l^n-1)}[e^{Lx(k^n-1)}-e^{-Lx}]$$

$$\leqslant\frac{K}{L^2}\|g-z\|\sum_{n=0}^{\infty}(\lambda kl)^n[1-e^{-La}][1-e^{-Lb}]+$$

$$+\frac{M}{L}\|g-z\|\sum_{n=0}^{\infty}(\lambda l)^n[1-e^{-Lb}]+\frac{N}{L}\|g-z\|\sum_{n=0}^{\infty}(\lambda k)^n[1-e^{-La}]$$
$$\leqslant\left[\frac{1}{L^2}\frac{K}{1-\lambda kl}+\frac{1}{L}\left(\frac{M}{1-\lambda l}+\frac{N}{1-\lambda k}\right)\right]\|g-z\|.$$

Since

$$\left[\frac{1}{L^2}\frac{K}{1-\lambda kl}+\frac{1}{L}\left(\frac{M}{1-\lambda l}+\frac{N}{1-\lambda k}\right)\right]<1$$

for $L>\frac{1}{2}\left\{\frac{M}{1-\lambda l}+\frac{N}{1-\lambda k}+\left[\left(\frac{M}{1-\lambda l}+\frac{N}{1-\lambda k}\right)^2+\frac{4K}{1-\lambda kl}\right]^{1/2}\right\}$,

we infer by the well-known Banach fixed-point theorem that equation (35) has a unique solution g^* defined in Δ. This solution is the limit of a uniformly convergent sequence $\{z_n\}$ of non-negative and continuous functions z_n defined by the relation

$$z_0(x,y)=0,\qquad (x,y)\in\Delta,$$
$$z_{n+1}(x,y)=Az_n(x,y),\qquad (x,y)\in\Delta,\ n=0,1,\ldots,$$

and, therefore, it is continuous, non-negative and non-decreasing, because z_n are so. This completes the proof of part (a).

We prove that the function g^* satisfies equation (36). Indeed, since g^* fulfils equation (35), we have

$$R(x,y)$$
$$\stackrel{\mathrm{df}}{=} g^*(x,y)-\lambda g^*(kx,ly)-K\int_0^x\int_0^y g^*(u,v)\,dudv-$$
$$-M\int_0^y g^*(x,v)\,dv-N\int_0^x g^*(u,y)\,du-h(x,y)$$
$$=g^*(x,y)-\lambda\Big[K\sum_{n=0}^{\infty}\lambda^n\int_0^{k^{n+1}x}\int_0^{l^{n+1}y}g^*(u,v)\,dudv+M\sum_{n=0}^{\infty}\lambda^n\int_0^{l^{n+1}y}g^*(k^{n+1}x,v)\,dv+$$
$$+N\sum_{n=0}^{\infty}\lambda^n\int_0^{k^{n+1}x}g^*(u,l^{n+1}y)\,du+\sum_{n=0}^{\infty}\lambda^n h(k^{n+1}x,\ l^{n+1}y)\Big]-$$
$$-K\int_0^x\int_0^y g^*(u,v)\,dudv-M\int_0^y g^*(x,v)\,dv-N\int_0^x g^*(u,y)\,du-h(x,y)$$
$$=g^*(x,y)-K\Big(\sum_{n=0}^{\infty}\lambda^{n+1}\int_0^{k^{n+1}x}\int_0^{l^{n+1}y}g^*(u,v)\,dudv+\int_0^x\int_0^y g^*(u,v)\,dudv\Big)-$$

$$-M\Big(\sum_{n=0}^{\infty}\lambda^{n+1}\int_0^{l^{n+1}y} g^*(k^{n+1}x, v)\,dv+\int_0^{y} g^*(x, v)\,dv\Big)-$$

$$-N\Big(\sum_{n=0}^{\infty}\lambda^{n+1}\int_0^{k^{n+1}x} g^*(u, l^{n+1}y)\,du+\int_0^{x} g^*(u, y)\,du\Big)-$$

$$-\Big(\sum_{n=0}^{\infty}\lambda^{n+1}h(k^{n+1}x, l^{n+1}y)+h(x, y)\Big)$$

$$= g^*(x, y)-K\sum_{n=0}^{\infty}\lambda^n\int_0^{k^nx}\int_0^{l^ny} g^*(u, v)\,du\,dv-M\sum_{n=0}^{\infty}\lambda^n\int_0^{l^ny} g^*(k^nx, v)\,dv-$$

$$-N\sum_{n=0}^{\infty}\lambda^n\int_0^{k^nx} g^*(u, l^ny)\,du-\sum_{n=0}^{\infty}\lambda^n h(k^nx, l^ny)\equiv 0,$$

and thus g^* is a solution of equation (36).

We prove that any measurable solution g of equation (36) satisfying the condition $0\leqslant g(x, y)\leqslant g^*(x, y)$, $(x, y)\in\Delta$, is a solution of equation (35).

Let $g_0(x, y)$, $(x, y)\in\Delta$, be a measurable solution of equation (36) satisfying the condition $0\leqslant g_0(x, y)\leqslant g^*(x, y)$, $(x, y)\in\Delta$. Put

$$\varphi_1(x, y) = K\int_0^x\int_0^y g_0(u, v)\,du\,dv+M\int_0^y g_0(x, v)\,dv+N\int_0^x g_0(u, y)\,du+h(x, y).$$

Now, for $(x, y)\in\Delta$, we have

$$(38)\qquad s(x, y)\overset{\mathrm{df}}{=}\sum_{n=0}^{\infty}\lambda^n\varphi_1(k^nx, l^ny)$$

$$= K\sum_{n=0}^{\infty}\lambda^n\int_0^{k^nx}\int_0^{l^ny} g_0(u, v)\,du\,dv+M\sum_{n=0}^{\infty}\lambda^n\int_0^{l^ny} g_0(k^nx, v)\,dv+$$

$$+N\sum_{n=0}^{\infty}\lambda^n\int_0^{k^nx} g_0(u, l^ny)\,du+\sum_{n=0}^{\infty}\lambda^n h(k^nx, l^ny) = Ag_0$$

$$\leqslant Kab\max_{\substack{0\leqslant x\leqslant a\\0\leqslant y\leqslant b}}|g_0(x, y)|\sum_{n=0}^{\infty}(\lambda kl)^n+Mb\max_{\substack{0\leqslant x\leqslant a\\0\leqslant y\leqslant b}}|g_0(x, y)|\sum_{n=0}^{\infty}(\lambda l)^n+$$

$$+Na\max_{\substack{0\leqslant x\leqslant a\\0\leqslant y\leqslant b}}|g_0(x, y)|\sum_{n=0}^{\infty}(\lambda k)^n+H(x, y)<\infty,$$

and it follows from Lemma 3 that the equation

$$(39)\qquad g(x, y) = \lambda g(kx, ly)+\varphi_1(x, y),\qquad (x, y)\in\Delta,$$

with

$$\varphi_1(x, y) = K\int_0^x\int_0^y g_0(u, v)\,dudv + M\int_0^y g_0(x, v)\,dv + \\ + N\int_0^x g_0(u, y)\,du + h(x, y), \qquad (x, y)\in \Delta,$$

has a unique solution in the class $0 \leqslant g(x, y) \leqslant (Ag_0)(x, y)$, $(Ag^*)(x, y) = g^*(x, y)$, and this solution is the function s defined by $s(x, y) = (Ag_0)(x, y)$.

Further, we put

$$\varphi_2(x, y) = K\int_0^x\int_0^y g^*(u, v)\,dudv + M\int_0^y g^*(x, v)\,dv + N\int_0^x g^*(u, y)\,du + h(x, y).$$

It is obvious that equation (39) with $\varphi_2(x, y)$ instead of $\varphi_1(x, y)$ also has a unique solution in the class $0 \leqslant g(x, y) \leqslant (Ag^*)(x, y) = g^*(x, y)$.

Now, from Lemma 4 it follows that the function $s(x, y) = (Ag_0)(x, y)$ is the unique solution of (39) in the class $0 \leqslant g(x, y) \leqslant g^*(x, y)$, $(x, y)\in \Delta$.

Because g_0 is also a solution of (39) in the class $0 \leqslant g(x, y) \leqslant g^*(x, y)$, $(x, y)\in \Delta$, we have $s(x, y) = g_0(x, y)\in \Delta$. Hence g_0 is a solution of (35) and, therefore, it is continuous.

Since each measurable solution of (36) in the class $0 \leqslant g(x, y) \leqslant g^*(x, y)$, $(x, y)\in \Delta$, is a solution of (35), the function g^* is the unique solution of (35), and g^* satisfies equation (36), we infer that the function g^* is the unique solution of (36) in the class pointed above. This completes the proof of part (b).

Now we prove that the function $g(x, y) \equiv 0$, $(x, y)\in \Delta$, is the unique measurable solution of the equation

$$g(x, y) = \lambda g(kx, ly) + K\int_0^x\int_0^y g(u, v)\,dudv + \\ + M\int_0^y g(x, v)\,dv + N\int_0^x g(u, y)\,du, \qquad (x, y)\in \Delta, \tag{40}$$

satisfying the condition $0 \leqslant g(x, y) \leqslant g^*(x, y)$, $(x, y)\in \Delta$.

Let $g_0(x, y)$ be a measurable solution of (40) fulfilling this condition. According to considerations of the proof of (b), we see that g_0 is a solution of equation (35) with $h = 0$, but since the only solution of that equation is $g(x, y) \equiv 0$, also $g_0(x, y) \equiv 0$.

Now (c) is implied by Remark 3.

Thus the proof of Lemma 6 is completed.

Remark 7. If the function ω does not depend on p, q, then assumption 2° of Lemma 6 can be replaced by $0 \leqslant \lambda kl < 1$.

Hence and from Lemma 2 and Theorem 2 we infer

THEOREM 8. *If assumption* H_1 *is satisfied, and*

1° *conditions* (31) *and* (32) *are satisfied,*

2° $H(x, y) \overset{\mathrm{df}}{=} \sum_{n=0}^{\infty} \lambda^n h(k^n x, l^n y) < \infty$, $(x, y) \in \Delta$, *where*

$$h(x, y) = \sup_{0 \leqslant \xi \leqslant x} \sup_{0 \leqslant \eta \leqslant y} |f(\xi, \eta, 0, 0, 0, 0)|$$

and H *is continuous for* $(x, y) \in \Delta$,

3° $0 \leqslant \lambda k < 1$, $0 \leqslant \lambda l < 1$,

4° $0 \leqslant k \leqslant 1$, $0 \leqslant l \leqslant 1$,

then there exists a unique and continuous solution $\bar{z}$ *of equation* (5) *with the properties*

$$|\bar{z}(x, y)| \leqslant g^*(x, y), \quad (x, y) \in \Delta,$$

$$|\bar{z}(x, y) - z_n(x, y)| \leqslant g_n(x, y), \quad (x, y) \in \Delta, \; n = 0, 1, \ldots,$$

where $g_0(x, y) = g^*(x, y)$, $(x, y) \in \Delta$, $g^*(x, y)$ *is defined as in Lemma* 6,

$$g_{n+1}(x, y) = \lambda g_n(kx, ly) + K \int_0^x \int_0^y g_n(u, v)\,du\,dv + M \int_0^y g_n(x, v)\,dv + $$
$$+ N \int_0^x g_n(u, y)\,du, \quad (x, y) \in \Delta, \; n = 0, 1, \ldots$$

The solution $\bar{z}$ *is unique in the class of functions satisfying the inequality* $|z(x, y)| \leqslant g^*(x, y)$, $(x, y) \in \Delta$.

Remark 8. Condition 2° of Theorem 8 is fulfilled if

$$|f(x, y, 0, 0, 0, 0)| \leqslant B(x+y), \quad (x, y) \in \Delta, \; B = \mathrm{const} \geqslant 0.$$

Theorem 5 implies the following

THEOREM 9. *If assumptions of Theorem* 8 (*except for* 2°) *are satisfied and if*

1° *functions* $\bar{z}$ *and* $\bar{p}$ *are solutions of equations* (5) *and* (19), *respectively,*

2° $H(x, y) \overset{\mathrm{df}}{=} \sum_{n=0}^{\infty} \lambda^n \psi(k^n x, l^n y) < \infty$, $(x, y) \in \Delta$, *where*

$$\psi(x, y) \geqslant \max\{ \sup_{0 \leqslant \xi \leqslant x} \sup_{0 \leqslant \eta \leqslant y} [|\bar{z}(\xi, \eta)| + |\bar{p}(\xi, \eta)|], \bar{h}(x, y)\}, \quad (x, y) \in \Delta,$$

3° $\bar{h}(x, y)$ *is defined by condition* 2° *of Theorem* 4, *and* H *is continuous in* Δ,

then

(a) *there exists a continuous, non-negative and non-decreasing solution* $\tilde{g}$ *of the equation*

$$g(x, y) = \lambda g(kx, ly) + K\int_0^x\int_0^y g(u, v)\,du\,dv + M\int_0^y g(x, y)\,dv + $$
$$+ N\int_0^x g(u, y)\,du + \psi(x, y), \qquad (x, y)\in \Delta,$$

(b) *the sequence* $\{\tilde{g}_n(x, y)\}$, $(x, y)\in \Delta$, *defined by*

$$\tilde{g}_0(x, y) = \tilde{g}(x, y), \qquad (x, y)\in \Delta,$$

$$\tilde{g}_{n+1}(x, y) = \lambda \tilde{g}_n(kx, ly) + K\int_0^x\int_0^y \tilde{g}_n(u, v)\,du\,dv + M\int_0^y \tilde{g}_n(x, v)\,dv + $$
$$+ N\int_0^x \tilde{g}_n(u, y)\,du + \bar{h}(x, y), \qquad (x, y)\in \Delta,\ n = 0, 1, \ldots,$$

has the limit $g^*(x, y)$ *in* Δ, *which is a continuous, non-negative and non-decreasing function satisfying* $g^*(x, y) \leqslant \tilde{g}(x, y)$, $(x, y)\in \Delta$,

(c) *the estimation* $|\bar{z}(x, y) - \bar{p}(x, y)| \leqslant g^*(x, y)$, $(x, y)\in \Delta$, *holds true.*

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