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SPANNING DISKS FOR DOMAINS WITH FREE FUNDAMENTAL GROUPS

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If a closed 2-manifold T in S^3 is locally polyhedral except at one point p and if A is a component of S^3-T, then there may or may not exist a disk D such that $\mathrm{Int}(D) \subset A$ and $\mathrm{Bd}(D)$ bounds in T a disk which contains p in its interior. If such a disk exists, we say that T can be ***spanned from A at p***, and we call this disk a ***spanning disk***. If T is wild from A at p, there must exist an $\varepsilon > 0$ such that each spanning disk D is of diameter greater than ε. The purpose of this paper is to show that if the fundamental group $\pi(A)$ of A is a free group of finite rank, then T can be spanned from A at p.

In [6], Lemma 1, p. 575, Harrold and Moise essentially showed that a 2-manifold T locally polyhedral except at one point p can be spanned from a component of S^3-T at p with a spanning disk of arbitrarily small diameter. It follows from the proof of Theorem I, p. 577, in [6], that T is therefore locally tame from this component at p.

To construct an example for which there does not always exist a spanning disk, we start with an unknotted polyhedral torus T' in S^3. Let A' denote a component of S^3-T'. Remove the interior of a disk in T' and sew back a disk (in $\mathrm{Cl}(A')$) which is wild at just an interior point p and which tapers to p as it runs once around the core of A' before entangling with itself. If we let T be the resulting torus and A be the component of S^3-T which is contained in A', then T cannot be spanned from A at p.

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In what follows, we use the symbols $I = [0, 1]$, and $N(K, \varepsilon)$ to denote the ε-neighborhood of K.

LEMMA. *Let T be a closed 2-manifold of genus h in S^3, A a component of S^3-T, and $p \in T$. Suppose that T is locally polyhedral except at p and wild from A at p. Let D_1 and D_2 be disjoint disks satisfying* (*for* $i = 1, 2$)

(1) $\mathrm{Int}(D_i) \subset S^3-\mathrm{Cl}(A)$,

(2) $\mathrm{Bd}(D_i) \subset T$,

(3) $\mathrm{Bd}(D_i)$ *bounds on* T *a disk* E_i *which contains* p *in its interior,*
(4) $E_1 \subset E_2$.

Let $T' = (T-E_2)\cup D_2$, *and let* C *be that component of* S^3-T' *which contains* A. *If* $\pi(A)$ *is a free group on the* h *generators* $x_1, x_2, \ldots, x_h$, *then the homomorphism* $i_*\colon \pi(A)\to\pi(C)$ *induced by the inclusion map* $i\colon A\to C$ *is an isomorphism.*

Proof. We first show that i_* is onto. Let $[f]\in\pi(C)$, where the brackets indicate the equivalence class which a loop f represents. We may assume that $f\colon I\to C$ is a simplicial map and that $f(I)\cap E_2$ consists of a finite number of points none of which are p. Let M be that component of $S^3-(E_2\cup D_2)$ which is contained in $S^3-\mathrm{Cl}(A)$. By [6], $\mathrm{Cl}(M)$ is a 3-cell. Therefore, f is homotopic in C to a loop $f_1\colon I\to C$ such that

$$f_1(I)\subset\big(A\cup\mathrm{Int}(E_2)\big)-p.$$

By [1], Theorem 1, p. 337, $T-p$ can be collared from A. That is, there is a homeomorphism

$$H\colon (T-p)\times I\to(T-p)\cup A$$

such that, for each $x\in T-p$,

$$H(x,0)=x \quad\text{and}\quad H(x,t)\in A \text{ for } t\neq 0.$$

Therefore, f_1 is homotopic in C to a loop $f_2\colon I\to A$. Then, in $\pi(C)$,

$$i_*([f_2])=[f_1]=[f].$$

Therefore i_* is onto.

We need to show that i_* is one-to-one. If X_1 and X_2 are open pathwise connected subsets of a space X such that $X=X_1\cup X_2$ and $X_1\cap X_2$ is non-empty and pathwise connected, then when applying the van Kampen theorem (see [3], p. 71), we adopt the notation

$$\pi(X_1)\mathop{*}_{G}\pi(X_2) \text{ for } \pi(X), \quad \text{where } G=\pi(X_1\cap X_2).$$

By [2], Lemma 1, p. 249, there is a pinched collar F attached to $\mathrm{Cl}(M)$ along the set E_2 such that $F\subset\mathrm{Cl}(A)$. Since F is a pinched 3-cell, $\mathrm{Int}(F)$ is homeomorphic to the interior of a solid torus. Therefore, $\pi\big(\mathrm{Int}(F)\big)=Z$, where Z denotes an infinite cyclic group. Since $F\cup\mathrm{Cl}(M)$ is homeomorphic to $\mathrm{Cl}(M)$, $F\cup\mathrm{Cl}(M)$ is a 3-cell. Therefore

$$\pi\Big(\mathrm{Int}\big(F\cup\mathrm{Cl}(M)\big)\Big)=1.$$

Letting

$$X=C-p,\quad X_1=A,\quad X_2=\mathrm{Int}\big(F\cup\mathrm{Cl}(M)\big),\quad\text{and}\quad G=\pi\big(\mathrm{Int}(F)\big),$$

we see from the van Kampen theorem that

$$\pi(C-p)=\pi(A)\mathop{*}_{G}\pi\Big(\mathrm{Int}\big(F\cup\mathrm{Cl}(M)\big)\Big)=(x_1,x_2,\ldots,x_h:)\mathop{*}_{Z}1=(x_1,x_2,\ldots,x_h:r),$$

where r denotes one relator. Since $\pi(C-p) = \pi(C)$, we have

$$\pi(C) = (x_1, x_2, \ldots, x_h : r).$$

We show that r must be a trivial relator. The *deficiency* of a finitely presented group is defined as the maximum, taken over all possible finite presentations of the group, of the integers $n-m$, where n is the number of generators and m the number of relators of each presentation. If we assume that r is not trivial, then, according to [4], Lemma 1.7, p. 207, the deficiency of $\pi(C)$ is $h-1$. It follows from [5], Theorem (1), p. 462, that C is homeomorphic to S^3-L, where L is a connected polygonal graph whose 1-dimensional Betti number is h. Examining a Dehn presentation of $\pi(S^3-L)$, we find that the deficiency of $\pi(S^3-L)$ and, therefore, of $\pi(C)$ is at least h, a contradiction. Hence

$$\pi(C) = (x_1, x_2, \ldots, x_h :).$$

Since $i_*: \pi(A) \to \pi(C)$ is onto and both $\pi(A)$ and $\pi(C)$ are free groups on h generators, it follows from the Hopfian property of free groups that i_* is one-to-one (see [7], Theorem 2.13, p. 109). This completes the proof that i_* is an isomorphism.

THEOREM. *With the same hypothesis as that of the Lemma, there exists a polyhedral disk D such that* $\mathrm{Int}(D) \subset A$ *and* $\mathrm{Bd}(D) = \mathrm{Bd}(E_1)$.

Proof. Since T is locally polyhedral except at p, there is a polyhedral annulus H such that

$$H \cap T = \mathrm{Bd}(E_1) \subset \mathrm{Bd}(H) \quad \text{and} \quad H - \mathrm{Bd}(E_1) \subset A.$$

Let $J = \mathrm{Bd}(H) - \mathrm{Bd}(E_1)$. Clearly, J can be shrunk to a point in $\pi(C)$. Since, according to the Lemma, i_* is one-to-one, J can be shrunk to a point in $\pi(A)$. Therefore, there exists a continuous map $F: I \times I \to A$ such that F on $\mathrm{Bd}(I \times I)$ is a homeomorphism onto J. Since

$$T \cap F(I \times I) = \emptyset,$$

$H \cup F(I \times I)$ is a Dehn disk in $A \cup \mathrm{Bd}(E_1)$. Let H' be a polyhedral annulus in H such that

$$H' \cap T = \mathrm{Bd}(E_1) \quad \text{and} \quad H' \cap F(I \times I) = \emptyset.$$

Choose

$$\varepsilon = \frac{1}{2} \operatorname{dist}\big(F(I \times I) \cup \mathrm{Cl}(H - H'), T\big).$$

It follows from Dehn's lemma (see [8], p. 1) that there exists a polyhedral disk D such that

$$\mathrm{Bd}(D) = \mathrm{Bd}(E_1), \quad H' \subset D, \quad D - H' \subset N\big(F(I \times I) \cup \mathrm{Cl}(H - H'), \varepsilon\big).$$

By construction, $\mathrm{Int}(D) \subset A$.

QUESTION. If T is a torus which is wild from a complementary domain A at just p and if $\pi(A)$ is a knot group different from Z, then can T be spanned from A at p? (**P 1020**)

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