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ON ALGEBRAIC RADICALS IN MOBS

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We recall* that a *mob* is a non-empty Hausdorff space S together with a continuous associative multiplication, denoted by juxtaposition, $(x, y) \to xy$. Let A be any subset of the mob S. The *algebraic radical* of A is defined to be the set $\{x \in S | x^k \in A$ for some integer $k \geqslant 1\}$ and is denoted by $\mathscr{R}(A)$. This set A is said to be *radically stable* if and only if $\mathscr{R}(A) = \mathscr{R}(\bar{A})$ holds. Obviously for any open subset A of S, A need not be radically stable. The purpose of this paper is to study some properties of the algebraic radicals of ideals in S. Our main result is:

Under some special conditions, any open ideal A of S can be radically stable without requiring that $\mathscr{R}(A)$ be closed.

Moreover, we will demonstrate that the notion of radical stability of an ideal in abelian mobs is useful: it gives a necessary and sufficient condition for the closure of a primary (prime) ideal to be primary (prime).

Throughout this paper, we use $\bar{C}$ to denote the closure of the set C and C' for the complement of C. Unless otherwise stated, S will be regarded as a compact abelian mob with zero. The reader is referred to [4] for terminology and notations.

1. Preliminaries. In this section, pertinent notations, definitions and properties of algebraic radicals of an abelian mob S (not necessarily compact) will be given. Most of them are well known results from ring theory which will be used later.

Notation. Let A be a subset of S.

$J(A) = A \cup AS$, that is, the smallest ideal containing A.

$\mathcal{J}_0(A)$ = the union of all ideals contained in A, that is, the largest ideal contained in A if there are any.

Definition 1.1. (1) A mob S with zero is said to be *0-prime* if and only if whenever $a, b \in S$, $ab = 0$, then $a = 0$ or $b = 0$.

(2) A mob S is said to be an *Ω-mob* if and only if for any two ideals I_1 and I_2 such that $I_1 \cap I_2 \neq \emptyset$, either $I_1 \subset I_2$ or $I_2 \subset I_1$.

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Definition 1.2. (1) An ideal P of S is said to be *prime* if and only if $ab \in P$ implies that $a \in P$ or $b \in P$.

(2) An ideal Q of S is said to be *primary* if and only if $ab \in Q$ implies that $a \in Q$ or there exists an integer $k \geqslant 1$ such that $b^k \in Q$.

(3) An ideal R of S is said to be *semi-prime* if and only if $a^2 \in R$ implies that $a \in R$.

(4) Let A, B be ideals of S. Define $A : B = \{x \in S | xB \subset A\}$ and call it the *ideal quotient* of A and B.

It is easy to see that $A : B$ is an ideal of S.

Definition 1.3. (1) An ideal A is *completely irreducible* (*irreducible*) if and only if whenever A is the intersection of a family (finite family) of ideals, then A is a member of the family.

(2) An ideal A is *w-reducible* if and only if A is the intersection of a family of open prime ideals containing A properly.

(3) An ideal A is *strongly reducible* (*weakly reducible*) if and only if A is the intersection of a finite family (infinite family) of ideals containing A properly.

Facts 1.4. The algebraic radicals of S have the following properties:
Let A, B be any subsets of S. Then

(1) $A \subset \mathscr{R}(A)$.
(2) $A^k \subset B$ implies that $\mathscr{R}(A) \subset \mathscr{R}(B)$ for any $k \geqslant 1$.
(3) $\mathscr{R}(\mathscr{R}(A)) = \mathscr{R}(A)$.
If A, B are ideals of S, then
(4) $\mathscr{R}(A)$ is an ideal of S.
(5) $\mathscr{R}(AB) = \mathscr{R}(A \cap B) = \mathscr{R}(A) \cap \mathscr{R}(B)$.
(6) If A is a primary ideal of S, then $\mathscr{R}(A)$ is a prime ideal of S which is the smallest prime ideal containing A.
(7) Let P, Q be ideals of S. Then Q is a primary ideal of S with $\mathscr{R}(Q) = P$ if and only if (i) $Q \subset P \subset \mathscr{R}(Q)$ and (ii) $ab \in Q$, $a \notin Q$ imply that $b \in P$.

The proofs of the above results are analogous to those in ring theory and we omit the proofs. The reader is referred to [6].

2. Prime and primary ideals. We are going to study, in this section, the prime and primary ideals of S, and, in particular, the algebraic radical of such ideals and their relationship.

Proposition 2.1. *An ideal N of S is a compact prime ideal if and only if S/N is an 0-prime mob.*

Proof. Suppose N is a compact prime ideal of S. Then N is closed in S. The Rees quotient S/N is formed by shrinking N to a single point with the quotient topology. S/N is a mob. Recall that the multiplication $*$ of S/N is defined in the following way:

$$a * b = ab \quad \text{if } a, b \text{ and } ab \text{ are in } S-N,$$

$$a * b = 0 \quad \text{if } ab \in N,$$

$$a * b = 0 \quad \text{if } a = 0 \text{ or } b = 0.$$

If $a*b = 0$, there are two possible cases: either (i) $a = 0$ or $b = 0$, or (ii) $ab \in N$. In case (ii), since N is prime, we have $a \in N$ or $b \in N$. This implies that $a = 0$ or $b = 0$ in S/N. Thus in either case $a = 0$ or $b = 0$. Hence S/N is 0-prime. Conversely, assuming that S/N is an 0-prime mob, since S/N is Hausdorff, the ideal N is closed in S and hence is compact. Suppose $x*y = 0$ in S/N, then we have $x = 0$ or $y = 0$ in S/N. This means that $x \in N$ or $y \in N$ in the mob S. Hence N is a compact prime ideal of S.

THEOREM 2.2. *Let A be an ideal of S such that $\mathscr{R}(A)$ is proper maximal in S. Then A is primary if and only if $S/\mathscr{R}(A)$ is an abstract completely 0-simple semigroup.*

Proof. Suppose A is a primary ideal of S; then $\mathscr{R}(A)$ is a prime ideal. As S is compact, it follows that $\mathscr{R}(A)$ is open by [4], p. 28. By theorem 2 of [3], p. 677, $\mathscr{R}(A)$ has the form $J_0(S-e)$ with e being a non--minimal idempotent of S. Therefore there exists $e^2 = e \notin \mathscr{R}(A)$. Now form the Rees quotient $S/\mathscr{R}(A)$. Clearly, $S/\mathscr{R}(A)$ is 0-simple ([4], p. 39) and contains e. Hence by [1], p. 655, $S/\mathscr{R}(A)$ is completely 0-simple. Conversely, suppose that $S/\mathscr{R}(A)$ is completely 0-simple. Then there exists an $e^2 = e \notin \mathscr{R}(A)$. Clearly, e is non-minimal. By the maximality of $\mathscr{R}(A)$, we have $\mathscr{R}(A) = J_0(S-e)$. By theorem 2 of [3], p. 677 again, $\mathscr{R}(A)$ is an open prime ideal of S. Now take $xy \in A$, then $xy \in \mathscr{R}(A)$. Thus $x \in \mathscr{R}(A)$ or $y \in \mathscr{R}(A)$. This implies that A is primary.

COROLLARY. *If E, the set of idempotents of S, is contained in a maximal proper ideal J of S, then J is a primary ideal of S.*

Proof. By [1], p. 655, S/J is either the zero semigroup of order two or else completely 0-simple. Since $E \subset J$, S/J contains no idempotents other than zero and hence S/J is the zero semigroup of order 2. Suppose $xy \in J, x \notin J, y \notin J$. Then $x \in S-J, y \in S-J$ in S/J. Since S/J is the zero semigroup of order 2, we have $y^2 = 0$, $x^2 = 0$ in S/J. This implies that $x^2 \in J, y^2 \in J$ in the mob S. Thus J is a primary ideal of S.

A. D. Wallace has proved the following result:

Let S be a compact mob (not necessarily abelian). Then each open prime ideal is completely irreducible, and each completely irreducible ideal is open by [5], p. 39.

One would naturally ask whether the irreducibility of an ideal Q in an abelian semigroup is a necessary and sufficient condition for Q to be primary. (This question was asked by A. D. Wallace in his lecture notes on topological semigroups, problem J6, p. 39 of [5]. We show here, by giving a counterexample, that the answer is negative.)

Example 2.3. Let S be an abelian semigroup consisting of four elements $\{0, a, b, c\}$ with multiplication table.

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	b	b
c	0	0	b	b

The sets $\{0, b\}, \{0, b, c\}, \{0, a, b\}$ are ideals of S. Now $\{0, b\} = \{0, b, c\} \cap \{0, a, b\}$. It is easily seen that $\{0, b\}$ is a primary ideal of S, but it is not irreducible. Thus we have shown that primary ideals in abelian mobs are not necessarily irreducible.

THEOREM 2.4. *If Q is an open semi-prime ideal of S, then Q is w-reducible.*

Before proving this theorem, we need the following two lemmas.

LEMMA 2.5. *Q is a semi-prime ideal if and only if $\mathscr{R}(Q) = Q$.*

Proof. If $\mathscr{R}(Q) = Q$, then it is easily seen that Q is semi-prime. Conversely, suppose that $Q \subsetneqq \mathscr{R}(Q)$, then there exists $a \in \mathscr{R}(Q)$ with $a \notin Q$. Let $k > 1$ be the minimal integer such that $a^k \in Q$. Suppose Q is semi-prime. Then k must be odd. Write $k = 2n+1 (n > 0)$. Since Q is an ideal, we infer that $a^{k+1} = a^k \cdot a \in Q$. Thus $a^{k+1} = a^{2n+2} = (a^{n+1})^2 \in Q$. Since Q is semi-prime, it follows that $a^{n+1} \in Q$. This contradicts the minimality of k. Hence $\mathscr{R}(Q) = Q$.

LEMMA 2.6. *Let Q be an open ideal of S, then $\mathscr{R}(Q) = \bigcap_\alpha P_\alpha$, where $\{P_\alpha\}$ are all the open prime ideals of S containing Q.*

Proof. Take $x \in \mathscr{R}(Q)$. Then there exists integer $k \geqslant 1$ such that $x^k \in Q \subseteq P_\alpha$ for all α. Since P_α is prime, $x \in P_\alpha$ for all α, that is, $x \in \bigcap_\alpha P_\alpha$. Hence $\mathscr{R}(Q) \subset \bigcap_\alpha P_\alpha$. Conversely, suppose that $\bigcap_\alpha P_\alpha \not\subset \mathscr{R}(Q)$. Then we can find an element y of $\bigcap_\alpha P_\alpha$ such that $y \notin \mathscr{R}(Q)$. We have $\overline{\{y, y^2, \ldots\}} = \Gamma(y) \subset J(y) \subset \bigcap_\alpha P_\alpha$.

Since $\Gamma(y)$ is compact, there exists an idempotent e such that $e \in \Gamma(y) \subset \bigcap_\alpha P_\alpha$ and $e \notin Q$. (For if $e \in Q$, then $\Gamma(y) \subset Q$. But $y \notin \mathscr{R}(Q)$.) Thus $J_0(S-e) \supset Q$. By theorem 2 of [3], p. 677, $J_0(S-e)$ is an open prime ideal of S. Therefore $J_0(S-e) \supset \bigcap_\alpha P_\alpha$. This implies that $e \notin \bigcap_\alpha P_\alpha$, a contradiction. Thus $\bigcap_\alpha P_\alpha \subset \mathscr{R}(Q)$.

By now, one can easily see that Theorem 2.4 is an immediate consequence of these two lemmas.

COROLLARY 1. *If* $|E| < \infty$, *any open ideal of* S *is semi-prime if and only if it is w-reducible.*

COROLLARY 2. $\mathscr{R}(Q)$ *is the smallest semi-prime ideal of* S *containing the ideal* Q.

COROLLARY 3. *Let* Q *be an open semi-prime ideal of* S. *If* B *is an ideal of* S *which is not contained in* Q, *then* B *contains an idempotent* e *with* $Se \not\subset Q$.

Proof. Let $b \in B - Q$. Consider the principal ideal $J(b)$ generated by b. Clearly, $J(b)$ is compact, $J(b) \subset B$, $J(b) \not\subset Q$. Now let $\mathscr{M}$ be the collection of all compact ideals $\{J_i\}_{i \in I}$ with the properties $J_i \subset B$, $J_i \not\subset Q$. By the same arguments as lemma 8 ([3], p. 676) we prove that there exists a minimal member J in $\mathscr{M}$ with $J \subset B$, $J \not\subset Q$. Now let $x \in J - Q$, and suppose $xJx \subset Q$. Since Q is semi-prime, by lemma 2.5 and lemma 2.6, $Q = \bigcap_a P_a$, where P_a are open prime ideals containing Q. As S is abelian, we have $J(x)^3 \subset xJx \subset Q \subset P_a$ for all a. This implies that $J(x) \subset P_a$ for all a. Hence $J(x) \subset \bigcap_a P_a = Q$, a contradiction. So we assert that $xJx \not\subset Q$. Since $xJx \subset J$ and J is minimal, we have $xJx = J$. Consequently, $x^n J x^n = J$ for all integers n. Thus $xJx = eJe = J$ with $e^2 = e \in \Gamma(x) \subset J$. Since $e \in J$, we have $eSe = Se \subset J \not\subset Q$.

THEOREM 2.7. *Let* F *be a closed ideal of* S *and let* $\mathscr{G} = \{$*open ideal* G_a *of* $S | G_a \supset F\}$. *Then* $F = \bigcap_a G_a$, $G_a \in \mathscr{G}$ *for all* a. *In other words,* F *is weakly reducible if the family* $\mathscr{G}$ *exists.*

Proof. Trivially, $F \subset \bigcap_a G_a$. To prove the converse containment, we only need to show that for any element $x \notin F$, $x \notin \bigcap_a G_a$. Since F is closed in S, it is compact. As S is compact Hausdorff, it is a regular space and hence there exists an open neighbourhood V containing F but excluding x. By the compactness of S, we have that $J_0(V)$ is an open ideal of S. Obviously, $F \subset J_0(V)$. Hence $J_0(V) \in \mathscr{G}$. Clearly, $x \notin J_0(V)$. This implies that $x \notin \bigcap_a G_a$.

COROLLARY. *If* S *satisfies the second axiom of countability, then* F *is a* $\mathscr{G}_\delta$*-ideal, that is,* F *can be expressed as a countable intersection of open ideals containing* F.

This is because compact and T_2 imply regular, and regular and second countability imply metrizable and every closed set in any metric space is G_δ.

THEOREM 2.8. *Let* S *be an abelian mob (not necessarily compact). If the algebraic radical of an ideal* A *is non-prime, then it is strongly reducible.*

Proof. Since $\mathscr{R}(A)$ is not prime, we can find elements x, y in S such that $xy \in \mathscr{R}(A)$ but $x \notin \mathscr{R}(A), y \notin \mathscr{R}(A)$. Consider $\mathscr{R}(A) : J(y) = \{z \in S | zJ(y) \subset \mathscr{R}(A)\}$. Then $\mathscr{R}(A) : J(y)$ is an ideal of S with $\mathscr{R}(A) \subset \mathscr{R}(A) : J(y)$. We claim that $\mathscr{R}(A) \neq \mathscr{R}(A) : J(y)$. In fact since $xy \in \mathscr{R}(A)$, we have that $xJ(y) = x(\{y\} \cup yS) = \{xy\} \cup xyS \subset \mathscr{R}(A)$. Thus $x \in \mathscr{R}(A) : J(y)$ but $x \notin \mathscr{R}(A)$. Now clearly $\mathscr{R}(A) \subset \big(\mathscr{R}(A) \cup J(y)\big) \cap \big(\mathscr{R}(A) : J(y)\big)$. On the other hand, if $t \in \big(\mathscr{R}(A) \cup J(y)\big) \cap \big(\mathscr{R}(A) : J(y)\big)$, then $tJ(y) \subset \mathscr{R}(A)$. If $t \notin \mathscr{R}(A)$, then we must have $t \in J(y)$. Hence $t^2 \in tJ(y) \subset \mathscr{R}(A)$. Since $\mathscr{R}(A)$ is semi-prime, we have $t \in \mathscr{R}(A)$. Hence we have shown that $\mathscr{R}(A) = \big(\mathscr{R}(A) \cup J(y)\big) \cap \big(\mathscr{R}(A) : J(y)\big)$ and hence $\mathscr{R}(A)$ is strongly reducible.

COROLLARY 1. *Let Q be an open primary ideal of the compact mob S with $\mathscr{R}(Q) = P$. If A is any closed ideal of S with $A \not\subset Q$, then $Q : A$ is an open primary ideal of S with $\mathscr{R}(Q : A) = P$.*

Proof. Since Q is open, Q' is closed and hence compact. A is also compact. If $x \in Q : A$, then $xA \cap Q' = \emptyset$. By the continuity of multiplication and the compactness of A, there exists a neighbourhood V of x such that $VA \cap Q' = \emptyset$. That is $VA \subset Q$. Hence $x \in V \subseteq Q : A$, that is, $Q : A$ is open. By 1.4 (7) and the fact that $(Q : A)A \subset Q$, we can obtain that (i) $Q : A \subset P \subset \mathscr{R}(Q : A)$ and (ii) $ab \in Q : A$, $a \notin Q : A$ imply that $b \in \mathscr{R}(Q : A)$. Hence, by 1.4 (7) again, $Q : A$ is an open primary ideal of S with $\mathscr{R}(Q : A) = P$.

COROLLARY 2. *If Q is a compact primary ideal of the compact mob S with $\mathscr{R}(Q) = P$ and if A is any ideal $\not\subset Q$, then $Q : A$ is a compact primary ideal of S with $\mathscr{R}(Q : A) = P$.*

In what follows, if the algebraic radical of an ideal A is an open prime ideal, then A is called a P-ideal of S.

PROPOSITION 2.9. *The set of all P-ideals of S forms a filter on S.*

This proposition follows immediately by observing that (1) Any finite intersection of P-ideals of S is a P-ideal. (2) Any arbitrary union of P-ideals of S is still a P-ideal.

Moreover, we remark that this union is a submob of S and is an open prime ideal of S.

Now, let e be an idempotent of a compact mob S. We say that an element $x \in S$ belongs to the idempotent e if e is the unique idempotent of $\Gamma(x) = \overline{\{x, x^2, \ldots\}}$. Let $B = \{x \in S | e_a \in \Gamma(x)\}$. We shall call it a B-class. Schwarz [4], p. 119, has proved that any compact abelian mob S can be written as the union of disjoint B-classes.

THEOREM 2.10. *Let A be a P-ideal of S. Then there exists at least one B-class which meets A but is disjoint from $S - \mathscr{R}(A)$.*

Proof. We may assume that there exists a B-class B_{a_0} such that $B_{a_0} \cap A \neq \emptyset$. Let $x \in B_{a_0} \cap A$. Then $x \in A$ and $x \in B_{a_0}$. Consider the principal ideal $J(x)$ generated by x. Clearly $J(x)$ is compact and $\{x, x^2, \ldots\} \subseteq J(x) \subset A$. Thus $\Gamma(x) \subseteq J(x)$. $\Gamma(x)$ has a unique idempotent which must be e_{a_0} since $x \in B_{a_0}$. Now, suppose there exists an element $y \in B_{a_0} \cap (S - \mathscr{R}(A))$. The element y also belongs to the idempotent e_{a_0}. But, since $y \in S - \mathscr{R}(A)$, and $\mathscr{R}(A)$ is prime, we have $\{y, y^2, \ldots\} \subseteq S - \mathscr{R}(A)$. As $\mathscr{R}(A)$ is open, $S - \mathscr{R}(A)$ is compact in S. It follows that $\overline{\{y, y^2, \ldots\}} = \Gamma(y) \subseteq S - \mathscr{R}(A)$. Therefore $e_{a_0} \in \Gamma(y) \subseteq S - \mathscr{R}(A)$. Therefore, $e_{a_0} \in \Gamma(y) \subseteq S - \mathscr{R}(A)$. This is impossible since A and $S - \mathscr{R}(A)$ are disjoint. Hence $B_{a_0} \cap (S - \mathscr{R}(A)) = \emptyset$.

COROLLARY. *Any P-ideal A contains exactly the same number of disjoint B-classes as* $\mathscr{R}(A)$. *More precisely,* $A \cap \{\bigcup_a B_a\} = \bigcup_a (A \cap B_a)$ *with* $B_a \subset P$.

3. Stability of algebraic radicals.

PROPOSITION 3.1. *If A is a subset of S with $\mathscr{R}(A)$ closed, and $x \in S$ is such that $A \subset xA$, then we have $\mathscr{R}(A) = \mathscr{R}(xA)$. In other words, the closed algebraic radical of A would not be expanded under any translation.*

Proof. By "Swelling lemma" ([2], p. 15), $A \subset xA \subset \bar{A}$. Hence $\mathscr{R}(A) \subset \mathscr{R}(xA) \subset \mathscr{R}(\bar{A})$. We only need to prove that $\mathscr{R}(\bar{A}) \subset \mathscr{R}(A)$. Since $A \subset \mathscr{R}(A)$, we have $\bar{A} \subset \overline{\mathscr{R}(A)} = \mathscr{R}(A)$. Consequently, $\mathscr{R}(\bar{A}) \subset \mathscr{R}(\mathscr{R}(A)) = \mathscr{R}(A)$. Thus we have obtained that $\mathscr{R}(xA) = \mathscr{R}(A)$.

THEOREM 3.2 (Main theorem). *Let A be an open ideal of S. Then A is radically stable if and only if $\mathscr{R}(\bar{A})$ does not contain any idempotent lying outside of A.*

In order to prove this theorem, the following lemma is crucial:

LEMMA 3.3. *Let A be any open ideal of S. If B is an ideal which is not contained in $\mathscr{R}(A)$, then B has an idempotent not in A.*

Proof. Since A is an ideal of S, so is $\mathscr{R}(A)$. As $B \not\subset \mathscr{R}(A)$, there exists an element $b \in B$ such that $b \notin \mathscr{R}(A)$. By the same method as theorem 2.10, we prove that there exists an idempotent $e^2 = e \in \Gamma(b) \subseteq J(b) \subset B$. Suppose on the contrary that $e \in A$. Then $K(b) = e\Gamma(b) \subseteq A$, where $K(b) = \bigcap_{n=1}^{\infty} \overline{\{b^i | i \geqslant n\}}$ ([3], p. 25). Since A is open, we have $b^n \in A$ for some integer $n \geqslant 1$. Thus $b \in \mathscr{R}(A)$ which is impossible.

Remark. For any compact abelian mob S and A a non-empty open subset of S, if B is a submob of S such that $B \not\subset \mathscr{R}(A)$, then $\bar{B}$ contains an idempotent which is not in $\mathscr{R}(A)$.

We are now ready to prove Theorem 3.2. As A is an ideal, so are $\bar{A}$ and $\mathscr{R}(\bar{A})$. For the necessity, we suppose that $\mathscr{R}(\bar{A}) \not\subset \mathscr{R}(A)$. Then, by our lemma, there exists an idempotent $e^2 = e \in \mathscr{R}(\bar{A})$, $e \notin A$. But we assume that such idempotent does not exist. Hence, $\mathscr{R}(\bar{A}) \subset \mathscr{R}(A)$. As $\mathscr{R}(A) \subset \mathscr{R}(\bar{A})$ always holds, we have $\mathscr{R}(A) = \mathscr{R}(\bar{A})$, that is, A is radically stable.

For the converse part, we assume that A is radically stable, that is, $\mathscr{R}(\bar{A}) = \mathscr{R}(A)$. Suppose there exists $e^2 = e \in \mathscr{R}(\bar{A})$. Then $e \in \mathscr{R}(A)$, so there exists $k \geqslant 1$ such that $e^k \in A$. Thus $e \in A$ and hence, $\mathscr{R}(A)$ contains no idempotents which are not in A. Our proof is complete.

COROLLARY 1. *Let A be any ideal of the mob S such that $\mathscr{R}(A)$ is open and properly contained in S. Then any ideal of S containing $\mathscr{R}(A)$ contains a compact group which is disjoint from A. Conversely, let G be a compact group in S such that G is disjoint from an open ideal A, and suppose that A contains all the other idempotents of S. Then $\mathscr{R}(A)$ is an open ideal of S disjoint from G.*

Proof. By corollary 3 of lemma 2.6, we have $eSe \not\subset \mathscr{R}(A)$ for some idempotent e. Now eSe is a compact submob of S with identity e. Consider $G_e = \{g \in eSe | gg^{-1} = e\}$. This is the maximal subgroup of eSe. It is known that G_e is a compact subgroup of eSe ([2], p. 13). We claim that $e \notin \mathscr{R}(A)$. For if $e \in \mathscr{R}(A)$, then $eSe \subset \mathscr{R}(A)$, a contradiction. Let us now suppose that $G_e \cap \mathscr{R}(A) \neq \emptyset$, then there exists $g \in G_e$ such that $g \in \mathscr{R}(A)$. Since $\mathscr{R}(A)$ is an ideal of S, $gg^{-1} = e \in \mathscr{R}(A)$, which is impossible. For the converse part, suppose $G \cap A = \emptyset$. Since G is a group, $g^k \in G$ for all $k \geqslant 1$, where $g \in G$. Hence $g^k \notin A$ for all $k \geqslant 1$. This implies that $g \notin \mathscr{R}(A)$. Thus $G \cap \mathscr{R}(A) = \emptyset$. As G and S are compact, $J_0(S-G)$ is an open ideal of S. Clearly, $\mathscr{R}(A) \subset J_0(S-G)$. Suppose that $J_0(S-G) \not\subset \mathscr{R}(A)$. Then by our lemma 3.3, there exists $e^2 = e \in J_0(S-G)$, $e \notin A$. This contradicts our assumption on A. Hence $\mathscr{R}(A) = J_0(S-G)$ and hence $\mathscr{R}(A)$ is an open ideal of S.

COROLLARY 2. *Let S be an Ω-mob. If A is an open ideal of S which is not radically stable and $\overline{\mathscr{R}(A)}$ is semi-prime, then $\mathscr{R}(\bar{A})$ is closed and has the form Se with $e^2 = e \notin \mathscr{R}(A)$.*

Proof. The non-radical stability of A implies that $\mathscr{R}(A) \subsetneq \mathscr{R}(\bar{A})$. By using the same method as lemma 8 in [3], p. 676, and our lemma 3.3, we can prove that there exists a minimal closed ideal M contained in $\mathscr{R}(\bar{A})$, but not contained in $\mathscr{R}(A)$. Moreover, M has the form Se with $e^2 = e \notin \mathscr{R}(A)$. Since S is compact, $\overline{Se \cap \mathscr{R}(A)} \neq \emptyset$. As S is Ω, it follows that $\mathscr{R}(A) \subset Se \subset \mathscr{R}(\bar{A})$. Hence $\overline{\mathscr{R}(A)} \subset Se$. Since $\overline{\mathscr{R}(A)}$ is semi-prime, we have $\mathscr{R}\big(\overline{\mathscr{R}(A)}\big) = \overline{\mathscr{R}(A)}$. Thus $\bar{A} \subset \overline{\mathscr{R}(A)}$ implies that $\mathscr{R}(\bar{A}) \subset \overline{\mathscr{R}(A)}$. We have, therefore, $\mathscr{R}(\bar{A}) = Se$ with $e^2 = e \notin A$.

COROLLARY 3. *Let A be an ideal which is radically stable in S. Then A is a primary ideal if and only if $\bar{A}$ is a primary ideal.*

Proof. We only need to observe that an ideal A is primary if and only if $\mathscr{R}(A)$ is prime.

Here we give two examples to demonstrate that, without radical stability, the closure of a prime (primary) ideal need not be prime (primary).

Example 3.3. Let S be the subset of the plane defined by formula $S = (\underline{[0,1]}\times 0)\cup(1\times\underline{[-1,1]})$ (see Fig. 1), where the underlined brackets denote the intervals, and define a commutative multiplication on S by:

$(x,0)*(1,v) = (x,0)$ for all points $x\in\underline{[a,b]}$, $v\in\underline{[c,d]}$.
$(x,0)*(y,0) = (xy,0)$ for all points $x,y\in\underline{[a,b]}$.
$(1,x)*(1,y) = (1,xy)$ for all points $x,y\in\underline{[b,c]}$.
$(1,x)*(1,y) = (1,-xy)$ for all points $x,y\in\underline{[b,d]}$.
$(1,x)*(1,y) = (1,0)$ if $x\in\underline{[b,d]}$, $y\in\underline{[b,c]}$ and vice versa.
Where xy is the usual product of x and y.

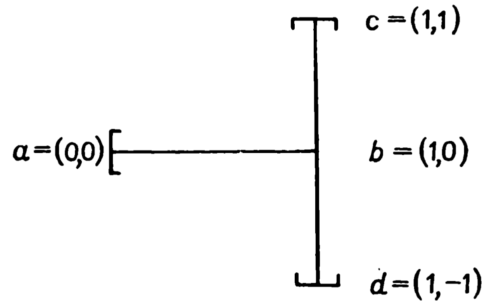

Fig. 1

Clearly, $\underline{[a,b)}$ is a prime ideal of S. Also $(1,1)*(1,-1) = (1,0)$ $\in\underline{[a,b]}$, but $(1,1)$, $(1,-1)$ are not points in $\underline{[a,b]}$. Hence, the closure of $\underline{[a,b)}$ is not a prime ideal of S.

Example 3.4. Let S be the subset of the plane defined by $S = (\underline{[0,1]}\times \underline{(-1,1)})\cup(1\times\underline{[1,-1]})$ (see Fig. 2) where the underlined brackets denote intervals, and define multiplication on S by:

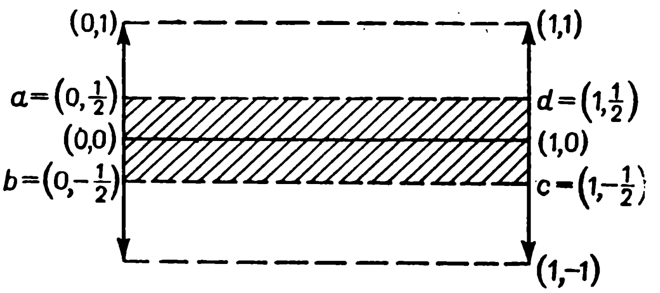

Fig. 2

$(x,y)*(u,v) = (xu,yv)$ for all points $(x,y),(u,v)$ in the upper half plane.

$(x,y)*(u,v) = (xu,-yv)$ for all points $(x,y),(u,v)$ in the lower half plane.

$(x,y)*(u,v) = (xu,0)$ if one of the points lies in the upper half plane and the other lies in the lower half plane.

Clearly, the rectangle $Q = \underline{(0,1)}\times(-\frac{1}{2}, \frac{1}{2})$ is a primary ideal of S, but the closure of Q is not primary.

Remark 1. Every ideal of the usual thread I is a primary ideal. By a *usual thread* we mean a semigroup topologically isomorphic to $[0,1]$ with its usual real multiplication. Obviously, the minimal ideal, $\{0\}$, of I is primary. Any non-minimal ideal of I has the form $[0,x)$ or $[0,x]$ for a fixed x in $(0,1]$ by [4], p. 84. To see that $[0,x)$ is primary, suppose $ab\in[0,x)$, $a\notin[0,x)$. Then $0\leqslant ab<x$, $x\leqslant a\leqslant 1$. Hence, $0\leqslant b<x/a$, $x/a\leqslant 1$. Thus $0\leqslant b<1$. Since x is fixed, there exists $k\geqslant 1$ such that $b^k<x$. As $[0,x)$ is radically stable, $[0,x]$ is also a primary ideal of I.

Remark 2. Every ideal of the min-thread I is prime. By a *minthread*, we mean a semigroup topologically isomorphic to $[0,1]$ with multiplication $x*y=\min(x,y)$. This remark is clear.

4. Concluding remarks. The definitions of reducibility and irreducibility of ideals can be generalized as follows: An ideal A is said to be *$\mathscr{R}$-irreducible* if A is reducible such that if $A=\bigcap_a A_a$, where A_a are ideals of S, then there exists at least one A_a such that $\mathscr{R}(A_a)=\mathscr{R}(A)$. If $\mathscr{R}(A_a)\neq\mathscr{R}(A)$ for all a, then A is said to be *$\mathscr{R}$-reducible*. The following example shows that $\mathscr{R}$-reducible ideals exist.

Example 4.1. Let S be the semigroup consisting of four elements $\{0,a,b,c\}$ such that $a^2=a$, $c^2=c$ and all other products are zero. Clearly, $\{0\}$, $\{0,a\}$, $\{0,c\}$ are ideals of S with $\{0\}=\{0,a\}\cap\{0,c\}$. But $\mathscr{R}(\{0\})=\{0,b\}$, $\mathscr{R}(\{0,a\})=\{0,a,b\}$, $\mathscr{R}(\{0,c\})=\{0,c,b\}$. Thus $\{0\}$ is $\mathscr{R}$-reducible.

The following facts are easily verified:

(1) Any algebraic radical of an ideal which is open and non-prime in the compact mob S is $\mathscr{R}$-reducible.

(2) If A is strongly reducible such that $\mathscr{R}(A)$ is a maximal proper ideal of S, then A is $\mathscr{R}$-irreducible.

(3) If a primary ideal is strongly reducible, then it is $\mathscr{R}$-irreducible.

It should be pointed out that, in general, a primary ideal Q and its associated prime ideal $\mathscr{R}(Q)$ are topologically unrelated. For instance, the statement "Q is compact if and only if $\mathscr{R}(Q)$ is compact" is not true. For Q is compact does not imply $\mathscr{R}(Q)$ is compact (cf. Remark 1 in section 3). Also $\mathscr{R}(Q)$ is compact does not imply Q is compact. For take $S=[0,1]$ with the multiplication $*$ defined by $x*y=\frac{1}{2}xy$, for all x,y in S. Then $Q=[0,\frac{1}{2})$ is a primary ideal of S which is not compact while $\mathscr{R}(Q)=[0,1]$.

Also "Q is connected if and only if $\mathscr{R}(Q)$ is connected" is not true. For take $S=[0,\frac{1}{2}]\cup[1,2]$. Define $x*y=\frac{1}{8}xy$ for all x,y in S. Then $[0,\frac{1}{2}]$ is a primary ideal of S, $\mathscr{R}([0,\frac{1}{2}])=S$ is disconnected. On the other hand, take $S = \{(x, y) | 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$. Define $(x, y) * (x', y') = (0, yy')$. Let $Q = \{(x, y) | x \in \{0, 1\},\ 0 \leqslant y \leqslant 1\}$. Then it can easily be checked that Q is a primary ideal of S. As $\mathscr{R}(Q) = S$, $\mathscr{R}(Q)$ is connected, however, Q itself is disconnected.

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