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ON GROUP RINGS OF ORDERED GROUPS

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Let G be a right-ordered group and let R be a commutative unital ring. It is well known [2] that absence of zero divisors in R forces the group ring RG to have only trivial units. In this note we prove that RG has trivial unit group if and only if R has no non-zero nilpotent elements and no non-trivial idempotents. We also prove that, when G is an ordered group, RG has no R-integral elements except for R if and only if R has no non-zero nilpotent elements. Finally, we offer a brief argument to show that if G is an abelian group such that the torsion subgroup of G is finite, then G is determined up to isomorphism by its rational group algebra QG. This result extends the work of Perlis and Walker [3], in which G was assumed to be finite.

1. Background information. In this section we shall describe the notation and, for the reader's convenience, recall relevant definitions.

Let R be a unital ring and let G be a group. Then the group ring RG of G over R is the free R-module with basis G and with multiplication extended R-bilinearly from the multiplication in G. It is a functor of both R and G. If J is an ideal of R and if $\bar{\alpha}$ is the image of α under the canonical homomorphism $R \to R/J$, then the mapping

$$RG \to (R/J)G, \quad \sum \alpha_g g \to \sum \bar{\alpha}_g g$$

is a surjective homomorphism with kernel JG. The kernel of the augmentation RG to R induced by collapsing G to 1 is called the *augmentation ideal* and will be denoted by $I(G)$. A unit $u \in RG$ is said to be *trivial* if $u = r \cdot g$ for some $r \in U(R)$, the unit group of R, and some $g \in G$. For $x = \sum x_g g \in RG$ the supporting subgroup of x is defined as the subgroup of G generated by $\operatorname{Supp} x = \{g \in G \mid x_g \neq 0\}$. A unit $u = \sum u_g g$ is called *normalised* if $\sum u_g = 1$ and a group basis in RG is said to be *normalised* if it consists of normalised units. Suppose that H is a subgroup of G and let T be a right transversal for H in G. Then RG is a free RH-module with T as a free basis. In particular, if $G = H_1 \times H_2$, then RG is a free RH_1-module with H_2 as a free basis, and since $sh = hs$ for any $s \in RH_1$ and for any $h \in H_2$, we have $R(H_1 \times H_2) = (RH_1)H_2$. A group G is said to be *ordered* (respectively, *right-ordered*) if the elements of G can be linearly ordered in a manner compatible with the group multiplication (respectively, with multiplication on the right). Any torsion-free nilpotent group is an ordered group and any group having a finite subnormal series with quotients which are torsion-free abelian is a right-ordered group (see [2]). After this digression into the terminology we now direct our attention to units and integral elements in RG.

2. Units and integral elements in group rings of ordered groups. The object of this section is to prove the following result:

THEOREM 1. (a) *Let R be a commutative unital ring and let G be a right-ordered group. Then the group ring RG has only trivial units if and only if R has no non-zero nilpotent elements and no non-trivial idempotents.*

(b) *Let R be an associative ring with 1 and let G be an ordered group. Then RG has no R-integral elements except for R if and only if R has no nilpotent elements.*

Proof. Let RG have only trivial units. If $0 \neq r \in R$ with $r^n = 0$ for some $n \geqslant 1$, then for any non-identity element g in G we have $(rg)^n = 0$ so that $1+rg$ is a non-trivial unit. If r is a non-trivial idempotent in R, then for any non-identity element g in G the unit $u = r\cdot 1+(1-r)g$ is non-trivial in RG (with $u^{-1} = r\cdot 1+(1-r)g^{-1}$). This proves the "only if" part of (a). Suppose now that R has no non-zero nilpotent elements and no non-trivial idempotents. Let $x = \sum x_g g$ and $y = \sum y_g g$ be elements in RG such that $xy = 1 = yx$. Then, by taking coefficients of 1 in $xy = 1$, we get

$$\sum x_g y_{g^{-1}} = 1, \tag{1}$$

and therefore

$$x_t y_{t^{-1}} \neq 0 \quad \text{for some } t \in G. \tag{2}$$

We claim that the result will hold provided the following is true:

$$x_g x_h = 0 \quad \text{whenever } g \neq h. \tag{3}$$

Indeed, if (3) is valid, then multiplication of (1) by x_t yields

$$x_t^2 y_{t^{-1}} = x_t \quad \text{and} \quad (x_t y_{t^{-1}})^2 = x_t y_{t^{-1}}.$$

Because of (2) this forces $x_t y_{t^{-1}} = 1$ and multiplication of this equality by x_g $(g \neq t)$ yields the desired result.

We are now left to prove (3). Suppose by way of contradiction that $x_g x_h \neq 0$ for some g, h in G such that $g \neq h$. Since $x_g x_h$ is not a nilpotent element, there exists a prime ideal P in R such that $x_g x_h \notin P$. Let $\bar{s}$ be the image of $s \in RG$ under the canonical homomorphism $RG \to (R/P)G$. Then

$$\bar{x}_g \neq 0,\ \bar{x}_h \neq 0 \text{ in } \bar{R} = R/P \quad \text{and} \quad \bar{x}\bar{y} = 1 = \bar{y}\bar{x} \text{ in } \bar{R}G.$$

Thus $\bar{R}G$ has a non-trivial unit, which is a contradiction because $\bar{R}$ is an integral domain (see [2]). This completes the proof of (a).

To prove (b), we first note that if $r^n = 0$ and $0 \neq r \in R$, then rg is R-integral for any $1 \neq g \in G$. Suppose, conversely, that R has no nilpotent elements and assume by way of contradiction that $\alpha \notin R$ is R-integral, i.e.

$$\alpha^n = r_1 \alpha^{n-1} + r_2 \alpha^{n-2} + \dots + r_n$$

for some $n \geqslant 2$ and some $r_i \in R$, $1 \leqslant i \leqslant n$. An easy induction argument shows that if s is the greatest element in $\operatorname{Supp}\alpha$ (respectively, t is the smallest element in $\operatorname{Supp}\alpha$), then s^k is the greatest element in $\operatorname{Supp}\alpha^k$ (respectively, t^k is the smallest element in $\operatorname{Supp}\alpha^k$). Suppose that $s > 1$. Then because of $s^n > s^i$ for any i, $0 \leqslant i < n$, it follows that $s^n \in \operatorname{Supp}\alpha^n$ but $s^n \notin \operatorname{Supp}\alpha^i$, $0 \leqslant i < n$, a contradiction. Thus $s \leqslant 1$. A similar argument shows that $t \geqslant 1$, and therefore $s = t = 1$, contrary to the assumption that $\alpha \notin R$. Consequently, it may be inferred that (b) also holds, thus completing the proof.

The corollary which follows is well known for the case where R is an integral domain (see [2]).

COROLLARY. *Let G be a right-ordered group and let R be a commutative unital ring without non-zero nilpotent elements and without non-trivial idempotents. Then RG determines G up to isomorphism.*

Proof. Let H be a group such that $RG \cong RH$. Without loss of generality we may assume that H is a normalised group basis of RG. Now, Theorem 1 may be used to infer that $H \subseteq G$, and since H is an R-basis of RG, we have $H = G$, as desired.

3. Isomorphisms of rational group algebras. It is well known [3] that if G is a finite abelian group, then G is determined up to isomorphism by its rational group algebra QG. The theorem which follows extends this result.

THEOREM 2. *Let G be an abelian group such that the torsion subgroup $T(G)$ of G is finite. Then G is determined up to isomorphism by QG.*

Proof. Let H be a group such that $QG \cong QH$. Without loss of generality we may assume that H is a normalised group basis of QG. It is clear that each element in $QT(H)$ is Q-integral. On the other hand, if α is a Q-integral element of QH, then by looking at the supporting subgroup of α we deduce that $\alpha \in Q(T(H_1) \times F)$, where H_1 and F are finite and torsion-free subgroups of H, respectively. Since $Q(T(H_1) \times F) = (QT(H_1))F$ and $QT(H_1)$ is semisimple, we infer from Theorem 1 that $\alpha \in QT(H_1) \leqslant Q(T(H))$. Thus $QT(H)$ consists of all Q-integral elements in QH, and therefore $QT(G) = QT(H)$. It follows from [3] that $T(G) \cong T(H)$. We now use a result due to Baer [1] to deduce that $G = T(G) \times F_1$ and $H = T(H) \times F_2$, where F_1 (respectively, F_2) is a torsion-free subgroup of G (respectively, of H). Since $QG \cdot I(T(G)) = QH \cdot I(T(H))$, we have

$$QF_1 \cong QG/QG \cdot I(T(G)) = QH/QH \cdot I(T(H)) \cong QF_2.$$

The desired result is now a consequence of the Corollary to Theorem 1.

It is known that if G is a finite p-group and if Q_p is the field of p-adic integers, then $Q_p G$ determines G up to isomorphism (see [4]). Therefore, the same proof as that of Theorem 2 gives the following useful companion to it.

THEOREM 3. *Let G be an abelian group such that the torsion subgroup of G is a finite p-group. Then G is determined up to isomorphism by its p-adic group algebra $Q_p G$.*

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