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A NOTE ON THE DIMENSION DIND

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Egorov and Podstavkin [1] introduced (following Arkhangelskiĭ) a definition of dimension Dind. The definition is both of inductive and covering type, namely

A1. Dind $X = -1$ iff $X = \emptyset$,

A2. Dind $X \leqslant N$ iff, for every finite open covering u, there exists a finite family v of mutually disjoint open sets being a refinement of u and such that Dind $(X - \bigcup\{V : V \in v\}) \leqslant N-1$.

It was proved in [1] that for perfectly normal spaces Dind coincides with dim. In our note we shall prove

THEOREM. *If a space X is normal, then*

$$\text{Dind } X = \text{Dind } \beta X,$$

where βX is the Čech-Stone compactification of X.

By a modification of the definition of Dind with the use of functionally open sets instead of open ones it can be proved in the same way that equality Dind X = Dind βX is valid also for completely regular spaces.

It is easy to prove that

(1) *if Y is a closed subspace of space X, then* Dind $Y \leqslant$ Dind X.

Let us assign to every open subset U of a normal space X an open set $E_X U$ in βX which satisfies the condition

(2) *$E_X U$ is the largest open subset V in βX such that $X \cap V = U$* (see [2], p. 269; originally in [3]).

The following properties of E_X will be used in the proof of the theorem (the closure operator $^{-}$ means always the closure in βX):

(3) $$E_X U = \beta X - \overline{(X-U)}.$$

(4) *If U and V are open in X, then* ([2], p. 269)

(4a) $$E_X(U \cup V) = E_X U \cup E_X V,$$

(4b) $$E_X(U \cap V) = E_X U \cap E_X V.$$

(5) *If U is open in X, then, for every open set V in βX such that $X \cap V = U$, we have $E_X U \subset \overline{V}$.*

(6) *If Y is a closed subset of a normal space X, then $\beta Y = \overline{Y}$* ([2], p. 130).

Proof of the Theorem. First we shall prove that

$$\text{(A)} \qquad \operatorname{Dind} X \leqslant \operatorname{Dind} \beta X.$$

Inequality (A) is valid if $\operatorname{Dind} \beta X = -1$. Suppose that it is valid for all normal spaces Y with $\operatorname{Dind} \beta Y \leqslant N-1$. Let the space X be such that $\operatorname{Dind} \beta X \leqslant N$.

Let $u = \{U_1, \ldots, U_n\}$ be an open covering of X. From (4a) it follows that $u^* = \{E_X U_1, \ldots, E_X U_n\}$ is an open covering of βX and from $\operatorname{Dind} \beta X \leqslant N$ it follows that there exists a family $w = \{W_1, \ldots, W_m\}$ of mutually disjoint open sets, which is a refinement of u^* such that

$$\text{(7)} \qquad \operatorname{Dind} (\beta X - \bigcup_{i=1}^{m} W_i) \leqslant N-1.$$

A family $v = \{V_1, \ldots, V_m\}$, where $V_i = X \cap W_i$ for $i = 1, \ldots, m$, is a refinement of covering u, and $V_i \cap V_j = \emptyset$ for $i \neq j$. Now to complete the proof of (A) it suffices to verify that

$$\operatorname{Dind} (X - \bigcup_{i=1}^{m} V_i) \leqslant N-1.$$

Since $V_i = X \cap W_i$, we infer from (2) that $W_i \subset E_X V_i$ for $i = 1, \ldots$ $\ldots, m$ and, in consequence,

$$\text{(8)} \qquad \beta X - \bigcup_{i=1}^{m} E_X V_i \subset \beta X - \bigcup_{i=1}^{m} W_i.$$

From (1), (7) and (8) we infer that

$$\text{(9)} \qquad \operatorname{Dind} (\beta X - \bigcup_{i=1}^{m} E_X V_i) \leqslant N-1.$$

From (3) and (4a) we have

$$\overline{X - \bigcup_{i=1}^{m} V_i} = \beta X - [\beta X - \overline{(X - \bigcup_{i=1}^{m} V_i)}] = \beta X - E_X \bigcup_{i=1}^{m} V_i = \beta X - \bigcup_{i=1}^{m} E_X V_i,$$

i. e.

$$\text{(10)} \qquad \overline{X - \bigcup_{i=1}^{m} V_i} = \beta X - \bigcup_{i=1}^{m} E_X V_i.$$

According to (6) and (10), we have

$$\text{(11)} \qquad \beta (X - \bigcup_{i=1}^{m} V_i) = \beta X - \bigcup_{i=1}^{m} E_X V_i.$$

By virtue of (9) and by the inductive assumption, equality (11) yields $\operatorname{Dind} (X - \bigcup_{i=1}^{m} V_i) \leqslant N-1$.

Now we prove that

(B) $$\operatorname{Dind} \beta X \leqslant \operatorname{Dind} X.$$

This is valid if $\operatorname{Dind} X = -1$. Let us assume that it is valid for all normal spaces Y with $\operatorname{Dind} Y \leqslant N-1$. Let X be a space such that $\operatorname{Dind} X \leqslant N$.

Let $u = \{U_1, \ldots, U_n\}$ be an open covering of βX. Since βX is a normal space, there exists an open covering $w = \{W_1, \ldots, W_n\}$ of X such that covering $\overline{w} = \{\overline{W}_1, \ldots, \overline{W}_n\}$ is a refinement of u. Let

(12) $$H_i = X \cap W_i \quad (i = 1, \ldots, n) \quad \text{and} \quad h = \{H_1, \ldots, H_n\}.$$

By virtue of $\operatorname{Dind} X \leqslant N$ there exists a family $g = \{G_1, \ldots, G_m\}$ of mutually disjoint open subsets of X which is a refinement of covering h such that

(13) $$\operatorname{Dind} (X - \bigcup_{i=1}^{m} G_i) \leqslant N-1.$$

Let

(14) $$V_i = E_X G_i \quad (i = 1, \ldots, m) \quad \text{and} \quad v = \{V_1, \ldots, V_m\}.$$

From the definition of v and w and from (5), (12) and (4b) we infer that the family v is a refinement of the covering u and $V_i \cap V_j = 0$ for $i \neq j$. It can be proved (see (10) and (11)) that

(15) $$\overline{X - \bigcup_{i=1}^{m} G_i} = \beta X - \bigcup_{i=1}^{m} E_X G_i = \beta X - \bigcup_{i=1}^{m} V_i$$

and, in consequence,

(16) $$\beta (X - \bigcup_{i=1}^{m} G_i) = \beta X - \bigcup_{i=1}^{m} V_i.$$

From (13), (16) and the inductive assumption we infer that

$$\operatorname{Dind} (\beta X - \bigcup_{i=1}^{m} V_i) \leqslant N-1.$$

This completes the proof of the theorem.

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