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CONNECTIVITY FUNCTIONS DEFINED ON I^n

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In [1] Brown constructed a connectivity function $f\colon I^2 \to I$ dense in I^3 without using the axiom of choice. The technique used in that construction is very tedious. In this paper, a very simple example is constructed without using the axiom of choice.

Let X and Y be topological spaces and let $f\colon X \to Y$ be a function. Then f is defined to be a connectivity function provided that if C is a connected subset of X, then the graph of f restricted to C, denoted by $f|C$, is a connected subset of $X \times Y$. The function f is defined to be peripherally continuous provided that if U is an open subset of X containing a point x of X and V is an open subset of Y containing $f(x)$, then there exists an open subset W of U containing x such that $f(\mathrm{bd}(W)) \subset V$, where $\mathrm{bd}(W)$ is the boundary of W.

EXAMPLE. Since a function from I^2 into I is a connectivity function if and only if it is peripherally continuous, we need only to construct a function which is peripherally continuous.

We defined the desired function $f\colon I^2 \to I$ as follows.

L_1: Let f be 0 on the boundary of I^2 and 1 at the center of I^2. Draw horizontal and vertical lines through the center which will divide I^2 into 4 small squares. Let f be linear on the edges of the small squares. The variation on the edges of the small squares is less than or equal to 1.

L_2: Let f be 0 or 1 in a checkerboard pattern at the center of the small squares of L_1. Divide the small squares of L_1 into 16 smaller squares. Let f be linear on the edges of these smaller squares. The variation on the edges of these squares is less than or equal to $\frac{1}{2}$. Continuing in this manner we have

L_{n+1}: Let f be 0 or 1 in a checkerboard pattern at the center of the small squares constructed in L_n. Divide the small squares of L_n into $(2(n+1))^2$ small squares. Let f be linear on the edges of these smaller squares. The variation on the edges of these squares is less than or equal to $1/(n+1)$.

Thus L_n is a grid and L_n is a subset of L_{n+1} for each n. By construction, f is peripherally continuous on $\bigcup_{n=1}^{\infty} L_n$.

Suppose x is in $I^2 - \bigcup_{n=1}^{\infty} L_n$. For each n, x is contained in the interior of some squares S_n such that, as $n \to \infty$, $S_n \to x$ and the variation of S_n tends to 0. Let y_n be in $\mathrm{bd}(S_n)$. Then $y_n \to x$. Let $f(x)$ be a cluster point of $f(y_n)$. Then f is peripherally continuous at x. By construction on L_n, f has a dense graph. Therefore, f has the desired properties.

In a paper presented at the 1968 conference on point set topology at the University of Houston, Cornette asked the question "Does there exist a space Y such that Y is the range of a connectivity function with domain I^n but not the range of a connectivity function with domain I^{n+1}, where n is any positive integer?" In [1] Brown answered the question for $n = 1$ but he was unable to show that the particular example constructed in that paper has this property. The authors of this paper have not been able to show that the example in this paper has this property. However, we prove the following theorems.

THEOREM 1. *If Y is the image of a connectivity function f: $I^n \to Y$, $n \geqslant 2$, then Y is not locally totally disconnected near any point where Y is a nondegenerate regular space.*

Proof. Suppose that, for some y in Y and some open subset V of Y containing y, V is totally disconnected. Let x be in $f^{-1}(y)$ and let U be an open subset of I^n containing x. Since f is peripherally continuous, there exists a connected open set W such that $x \in W \subset U$, $\mathrm{bd}(W)$ is connected, and $f(\mathrm{bd}(W)) \subset V$. Since $f(\mathrm{bd}(W))$ is connected, it must contain only one point, say z. Let C be a component of $f^{-1}(z)$ such that $\mathrm{bd}(W) \subset C$. Now C is closed. Let y be in $I^n - C$. Let $d_1 = d(y, C)$ and let w be in C such that $d(y, w) = d_1$. Now $f(w) = z$. Let d_2 be the diameter of C. Let U_0 be a spherical neighborhood with center w and radius less than the minimum of $\frac{1}{4}d_1$ and $\frac{1}{4}d_2$. Now there exists a connected open set U_1 such that $w \in U_1 \subset U_0$, $\mathrm{bd}(U_1)$ is connected, and $f(\mathrm{bd}(U_1)) \subset V$. Since C contains points in U_1 and not in U_1, $C \cap \mathrm{bd}(U_1) \neq \emptyset$. Since $C \cup \mathrm{bd}(U_1)$ is connected, $f(\mathrm{bd}(U_1)) = z$. Now $C \cup \mathrm{bd}(U_1)$ is a connected subset of $f^{-1}(z)$ which contains a point on the segment from w to y not belonging to C. This is a contradiction. Therefore Y is not locally totally disconnected near any point.

THEOREM 2. *Let f: $I^2 \to I$ be continuous and onto. Let g: $I \to Y$ be any function such that $g \circ f$: $I^2 \to Y$ is a connectivity function where Y is a continuum. Then g is continuous except perhaps at 0 or 1.*

Proof. Choose any a in the open interval (0, 1). Suppose a_n converges to a but $g(a_n)$ converges to b, where $g(a) \neq b$. In fact, we may assume that each a_n and a are in the open interval $(\varepsilon, 1-\varepsilon)$ for some $\varepsilon > 0$. Since f is continuous, $f^{-1}(a_n)$ for each n separates $f^{-1}([0, \varepsilon])$ from $f^{-1}([1-\varepsilon, 1])$. Let C_1 be a component of $f^{-1}([0, \varepsilon])$ and let C_2 be a component of $f^{-1}([1-\varepsilon, 1])$. Now, for each n, a certain component of each $f^{-1}(a_n)$ separates C_1 from C_2. Call these components K_n. Let A_1 and A_2 be two disjoint line segments from points of C_1 to points of C_2. Then each K_n intersects A_1 and A_2.

Now let y be a limit point of z_n in K_n for each n. Then

$$f(y) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} a_n = a.$$

Since y cannot lie in both A_1 and A_2, suppose y is not in A_1. Let

$$K = \Big(\bigcup_{n=1}^{\infty} K_n\Big) \cup A_1 \cup \{y\}$$

and K be connected. Since A_1 is closed and y is not in A_1, $(y, g(a))$ is not a limit point of $g \circ f | A_1$. Since $g(a_n)$ converges to b, $(y, g(a))$ is not a limit point of $g \circ f | \bigcup_{n=1}^{\infty} K_n$. Thus $(y, g(a))$ is not a limit point of $g \circ f | K$, and hence $g \circ f$ is not a connectivity function. This is a contradiction. Hence $g(a_n)$ converges to $g(a)$ and g is continuous except perhaps at 0 or 1.

EXAMPLE. Let f: $I^2 \to I$ be the x-projection and let g: $I \to I$ be defined by $g(x) = x$ if $0 < x \leqslant 1$ and $g(0) = 1$. Then f is continuous and g is continuous except at 0. Let L be the line segment in I^2 joining $(0, 0)$ and $(\frac{1}{4}, \frac{1}{4})$. Then $g \circ f | L$ is not connected. Thus $g \circ f$ is not a connectivity function.

QUESTION. Can Theorem 2 be strengthened to apply to f being a connectivity function? **(P 1343)**

Remarks. It is worth noting the following remarks made by the referee.

1. Theorem 1 may be formulated more generally by replacing I^n with a locally peripherally connected polyhedron [3].

2. Theorem 2 may be formulated more generally by replacing I^2 with a unicoherent, cyclicly connected Peano continuum. A space X is cyclicly connected iff every two points of X belong to a simple closed curve in X.

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