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ON CONTINUOUS FUNCTIONS AND THE APPROXIMATE SYMMETRIC DERIVATIVES

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Filipczak [2] constructed a continuous real-valued function of one real variable which has a symmetric derivative at no point on the real line. In fact, this function has the property that at each point its lower symmetric derivative is $-\infty$ and its upper symmetric derivative is $+\infty$. Letting C represent the metric space of all continuous real-valued functions on $[0,1]$ with the usual metric

$$\varrho(f,g) = \max_{x\in[0,1]}\{|f(x)-g(x)|\}, \quad f,g\in C,$$

Kostyrko [3] subsequently used this result to show that the collection of functions in C which have either an upper symmetric derivative less than $+\infty$ or a lower symmetric derivative greater than $-\infty$ at at least one point in $(0,1)$ is of the first category in C.

The purpose of the present note is to observe that the same results hold if one examines the approximate symmetric derivative instead of the ordinary symmetric derivative. It will be shown that the construction of Filipczak yields a function which has an approximate symmetric derivative nowhere (Theorem 1). Then, using an approach similar to that of Kostyrko, it will be verified that the collection of functions in C which have either an approximate upper symmetric derivative less than $+\infty$ or an approximate lower symmetric derivative greater than $-\infty$ at at least one point in $(0,1)$ is of the first category in C (Theorem 2).

Definitions of terms used in this article are as found in [4]. The notation $|G|$ will be used to denote the Lebesgue measure of a measurable set G.

We begin by examining the basic function $f_{a,\beta}$ as defined in [2].

LEMMA. *Let a and β be any two positive numbers. There exists a continuous function $f_{a,\beta}$ defined on the real line which satisfies the following conditions:*

(1) *$f_{a,\beta}$ is periodic of period $7a$.*

(2) *$|f_{a,\beta}(x)| \leqslant \beta$ for all x.*

(3) *$|f_{a,\beta}(x_1) - f_{a,\beta}(x_2)| \leqslant (\beta/a)|x_1 - x_2|$ for all x_1, x_2.*

(4) *For each real number x, there exist closed intervals $H = H(x)$ and $K = K(x)$ such that*

(4a) *$H \subset [a/2, 13a/2]$ and $K \subset [a/2, 13a/2]$.*

(4b) *$|H| = |K| = a/2$.*

(4c) *If $h \in H$, then*

$$\frac{f_{a,\beta}(x+h) - f_{a,\beta}(x-h)}{2h} \leqslant \frac{-\beta}{26a}.$$

(4d) *If $k \in K$, then*

$$\frac{f_{a,\beta}(x+k) - f_{a,\beta}(x-k)}{2k} \geqslant \frac{\beta}{26a}.$$

Proof. As in [2], we set

$$f_{a,\beta}(x) = f_{a,\beta}(x+7na) = \begin{cases} \beta x/a & \text{for } x \in [0, a], \\ \beta(2a - x)/a & \text{for } x \in [a, 3a], \\ \beta(x-4a)/a & \text{for } x \in [3a, 4a], \\ 0 & \text{for } x \in [4a, 7a]. \end{cases}$$

Clearly, $f_{a,\beta}$ is continuous and satisfies conditions (1), (2) and (3). In order to verify that (4) holds, we consider the following four cases:

$$0 \leqslant x \leqslant 2a, \tag{5}$$

$$2a \leqslant x \leqslant 4a, \tag{6}$$

$$4a \leqslant x \leqslant 6a, \tag{7}$$

$$6a \leqslant x \leqslant 7a. \tag{8}$$

For the remainder of this proof, let $f = f_{a,\beta}$.

Let x be a number satisfying (5). Then we will set

$$H = H(x) = [5a/2 - x, 3a - x] \quad \text{and} \quad K = K(x) = [4a + x, 9a/2 + x].$$

Then H and K, clearly, satisfy (4a) and (4b). Let $h \in H$, and note that $5a/2 \leqslant x+h \leqslant 3a$ and $-3a \leqslant x-h \leqslant 3a/2$. Consequently,

$$\frac{f(x+h) - f(x-h)}{2h} \leqslant \frac{f(x+h)}{2h} \leqslant \frac{-\beta}{4h} \leqslant \frac{-\beta}{26a},$$

i. e., (4c) holds. Next, let $k \in K$ and note that $4a \leqslant x+k \leqslant 17a/2$ and $-9a/2 \leqslant x-k \leqslant -4a$. Consequently,

$$\frac{f(x+k)-f(x-k)}{2k} \geqslant \frac{-f(x-k)}{2k} \geqslant \frac{\beta}{4k} \geqslant \frac{\beta}{26a},$$

i. e., (4d) holds.

Next suppose that x satisfies (6), and then set

$$H = H(x) = [x-3a/2,\ x-a] \text{ and } K = K(x) = [8a-x,\ 17a/2-x].$$

Then (4a) and (4b) are satisfied, and if $h \in H$, then $5a/2 \leqslant x+h \leqslant 7a$ and $a \leqslant x-h \leqslant 3a/2$. So

$$\frac{f(x+h)-f(x-h)}{2h} \leqslant \frac{-f(x-h)}{2h} \leqslant \frac{-\beta}{4h} \leqslant \frac{-\beta}{26a},$$

i. e., (4c) is satisfied. If $k \in K$, then $8a \leqslant x+k \leqslant 17a/2$ and $-9a/2 \leqslant x-k \leqslant 0$, and whence

$$\frac{f(x+k)-f(x-k)}{2k} \geqslant \frac{f(x+k)}{2k} \geqslant \frac{\beta}{4k} \geqslant \frac{\beta}{26a},$$

i. e., (4d) is satisfied.

In a similar fashion the reader can verify that if x satisfies (7), we can take

$$H(x) = [10a-x,\ 21a/2-x] \quad \text{and} \quad K(x) = [x-7a/2,\ x-3a],$$

and, finally, if x satisfies (8), we can take

$$H(x) = [x-a,\ x-a/2] \quad \text{and} \quad K(x) = [15a/2-x,\ 8a-x].$$

This will then complete the proof.

THEOREM 1. *There is a continuous function, defined everywhere on the real line, which has at every point neither a finite nor an infinite approximate symmetric derivative. More precisely, this function has an approximate lower symmetric derivative* $-\infty$ *and an approximate upper symmetric derivative* $+\infty$ *at every point.*

Proof. Let $0 < a < b < 1$, where the actual values for a and b will be fixed later. Then, following the procedure in [2], for each natural number n, set

$$f_n(x) = f_{a^n, b^n}(x), \tag{9}$$

and then

$$f(x) = \sum_{n=1}^{\infty} f_n(x). \tag{10}$$

As indicated in [2], this function is continuous.

Now, let x be an arbitrary real number and n a natural number. From the lemma we know there exist closed intervals

$$H_n = H_n(x) \subset [a^n/2,\ 13a^r/2] \quad \text{and} \quad K_n = K_n(x) \subset [a^n/2,\ 13a^n/2]$$

such that $|H_n| = |K_n| = a^n/2$,

$$\frac{f_n(x+h)-f_n(x-h)}{2h} \leqslant \frac{-1}{26}\left(\frac{b}{a}\right)^n \quad \text{for } h \in H_n,$$

and

$$\frac{f_n(x+k)-f_n(x-k)}{2k} \geqslant \frac{1}{26}\left(\frac{b}{a}\right)^n \quad \text{for } k \in K_n.$$

So, if $h \in H_n$, then

$$\begin{aligned}
\frac{f(x+h)-f(x-h)}{2h} &= \sum_{m=1}^{n-1} \frac{f_m(x+h)-f_m(x-h)}{2h} + \frac{f_n(x+h)-f_n(x-h)}{2h} + \\
&\qquad + \sum_{m=n+1}^{\infty} \frac{f_m(x+h)-f_m(x-h)}{2h} \\
&\leqslant \frac{-1}{26}\left(\frac{b}{a}\right)^n + \sum_{m=1}^{n-1} \frac{f_m(x+h)-f_m(x-h)}{2h} + \\
&\qquad + \frac{1}{2h} \sum_{m=n+1}^{\infty} [|f_m(x+h)| + |f_m(x-h)|] \\
&\leqslant \frac{-1}{26}\left(\frac{b}{a}\right)^n + \sum_{m=1}^{n-1} \left(\frac{b}{a}\right)^m + \frac{1}{a^n} \sum_{m=n+1}^{\infty} 2b^m \\
&= \frac{-1}{26}\left(\frac{b}{a}\right)^n + \frac{(b/a)^n - b/a}{b/a-1} + \frac{1}{a^n}\,\frac{2b^{n+1}}{1-b} \\
&\leqslant \left(\frac{b}{a}\right)^n \left(\frac{-1}{26} + \frac{a}{b-a}\,\frac{2b}{1-b}\right).
\end{aligned}$$

If we now fix $a = 0.0001$ and $b = 0.01$, we are assured that this last expression is less than or equal to -10^{2n-3}. Analogous calculations yield that if $k \in K_n = K_n(x)$, then

$$\frac{f(x+k)-f(x-k)}{2k} \geqslant 10^{2n-3}.$$

To finish the argument, let p be any positive number, and choose an N so that $10^{2N-3} > p$. Then, for any $n > N$, there is an interval $H_n = H_n(x)$ such that $H_n \subset [a^n/2,\ 13a^n/2]$, $|H_n| = a^n/2$, and, for each $h \in H_n$,

$$\frac{f(x+h)-f(x-h)}{2h} \leqslant -10^{2n-3} < -p$$

and

$$\frac{|H_n \cap [-13a^n/2,\ 13a^n/2]|}{13a^n} = \frac{1}{26}.$$

So the set

$$\left\{y\colon \frac{f(x+y)-f(x-y)}{2y} < -p\right\}$$

has the outer upper density at least $1/26$ at zero, and whence the approximate lower symmetric derivative at x is $-\infty$. Analogously, we conclude that at zero the set

$$\left\{y\colon \frac{f(x+y)-f(x-y)}{2y} > p\right\}$$

has the outer upper density at least $1/26$ by using the intervals K_n, and whence the approximate upper symmetric derivative at x is $+\infty$. So the proof is complete.

For notational purposes in the next theorem, we set

$$I_f(x, h) = \frac{f(x+h)-f(x-h)}{2h}$$

for $f \in C$, and x, $x-h$, $x+h$ all in $(0, 1)$.

THEOREM 2. *Let M be the collection of all functions $f \in C$ having the property that, for every $x \in (0, 1)$,*

$$\limsup_{h\to 0} \operatorname{ap} I_f(x, h) = +\infty \quad \textit{and} \quad \liminf_{h\to 0} \operatorname{ap} I_f(x, h) = -\infty.$$

Then the set $N = C \setminus M$ is of the first category in C.

Proof. Let

$$N_1 = \{f \in C\colon \exists x \in (0,1) \text{ such that } \limsup_{h\to 0} \operatorname{ap} I_f(x, h) < +\infty\}$$

and

$$N_2 = \{f \in C\colon \exists x \in (0,1) \text{ such that } \liminf_{h\to 0} \operatorname{ap} I_f(x, h) > -\infty\}.$$

Then $N = N_1 \cup N_2$. It will then suffice to show that both N_1 and N_2 are of the first category in C. We do this, first, for N_1.

For each $f \in C$, $x \in (0, 1)$, and $d > 0$, let $E(f, x, d) = \{h: I_f(x, h) < d\}$. Then, for each $n = 2, 3, \ldots$, set

$$Q_n = \left\{ f \in C: \exists x \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \text{ such that } |E(f, x, n) \cap [-b, b]| \geqslant \frac{58}{30} b, \text{ whenever } b < \frac{1}{n} \right\}.$$

Note that $N_1 \subset \bigcup_{n=2}^{\infty} Q_n$. We actually show that $\bigcup_{n=2}^{\infty} Q_n$ is of the first category in C. More precisely, we show that each Q_n is a closed nowhere dense set in C.

To this end fix an n, and let $f \in \bar{Q}_n$ ($\bar{Q}_n$ denotes the closure of Q_n). There is a sequence of functions $\{f_k\}_{k=1,2,\ldots} \subset Q_n$ such that $\varrho(f_k, f) \to 0$. For each $k = 1, 2, \ldots$, there exists an $x_k \in [1/n, 1-1/n]$ such that

$$|E(f_k, x_k, n) \cap [-b, b]| \geqslant \frac{58}{30} b \quad \text{whenever } b < \frac{1}{n}.$$

We can, clearly, assume that

$$\lim_{k\to\infty} x_k = x_0 \in [1/n, 1-1/n].$$

Let

$$E_0 = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E(f_k, x_k, n).$$

If $b < 1/n$, we have

$$\begin{aligned} |E_0 \cap [-b, b]| &= \lim_{m\to\infty} \left| \bigcup_{k=m}^{\infty} E(f_k, x_k, n) \cap [-b, b] \right| \\ &\geqslant \lim_{m\to\infty} |E(f_m, x_m, n) \cap [-b, b]| \geqslant \frac{58}{30} b. \end{aligned}$$

Next we show that $E_0 \subset E(f, x_0, n)$. Let $h \in E_0$. Suppose $\varepsilon > 0$. According to the Arzelà-Ascoli theorem (see, for instance, [1], p. 191), there exists a $\delta > 0$ such that $|x - x'| < \delta$ implies

$$|f_k(x) - f_k(x')| < \frac{\varepsilon |h|}{2} \quad \text{for all } k = 1, 2, \ldots$$

Next notice that

$$\lim_{k\to\infty} I_{f_k}(x_0, h) = I_f(x_0, h).$$

Consequently, we can choose a natural number $m > 1$ such that $k \geqslant m$ implies

$$|I_{f_k}(x_0, h) - I_f(x_0, h)| < \varepsilon/2 \quad \text{and} \quad |x_k - x_0| < \delta.$$

Since $h \in E_0$, there exists a $k \geqslant m$ such that $h \in E(f_k, x_k, n)$. So, for this k, we have $I_{f_k}(x_k, h) \leqslant n$, and

$$|I_f(x_0, h) - I_{f_k}(x_k, h)| \leqslant |I_f(x_0, h) - I_{f_k}(x_0, h)| + |I_{f_k}(x_0, h) - I_{f_k}(x_k, h)|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So $I_f(x_0, h) < n + \varepsilon$, and since this holds for each $\varepsilon > 0$, we conclude that $I_f(x_0, h) \leqslant n$, i.e., $h \in E(f, x_0, n)$. Then $E_0 \subset E(f, x_0, n)$, and whence $f \in Q_n$, i.e., Q_n is closed.

Next we show that Q_n is nowhere dense in C. Let p be a polynomial, $\varepsilon > 0$, and $B(p, \varepsilon) = \{g \in C\colon \varrho(g, p) < \varepsilon\}$. We show that $B(p, \varepsilon) \cap (C \setminus Q_n) \neq \emptyset$. Suppose p satisfies the Lipschitz condition with constant L. Now, let f represent the function constructed in Theorem 1. Then, as seen in the proof of that theorem, for each $x \in (0, 1)$, there exists a positive number b (which depends on x) with $b < 1/n$ such that

$$\left|E\left(f, x, \frac{L+n}{\eta}\right) \cap [-b, b]\right| \leqslant \left(1 - \frac{1}{26}\right) 2b < \frac{58}{30} b,$$

$$\text{where } \eta = \frac{\varepsilon}{2\|f\|} \text{ and } \|f\| = \max_{x \in [0,1]} \{|f(x)|\}.$$

Then $p + \eta f \in B(p, \varepsilon)$, and, for each $h \in E(p + \eta f, x, n) \cap [-b, b]$, we have

$$n \geqslant I_{p+\eta f}(x, h) = I_p(x, h) + \eta I_f(x, h) \geqslant -L + \eta I_f(x, h),$$

i.e., $I_f(x, h) \leqslant (L+n)/\eta$, implying that h belongs to $E\big(f, x, (L+n)/\eta\big) \cap$ $\cap\ [-b, b]$. Consequently, we have

$$E(p + \eta f, x, n) \cap [-b, b] \subset E\big(f, x, (L+n)/\eta\big) \cap [-b, b],$$

and so

$$|E(p + \eta f, x, n) \cap [-b, b]| < \frac{58}{30} b.$$

Hence $p + \eta f \notin Q_n$. So Q_n is nowhere dense. Thus $\bigcup_{n=2}^{\infty} Q_n$ is of the first category in C, implying that N_1 is also. An analogous reasoning shows that N_2 is likewise of the first category in C, and the proof is complete.

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