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# ON REPRESENTATION OF FUNCTIONALS OF LOCAL TYPE BY DIFFERENTIAL FORMS

BY

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**1. Introduction.** The problems which we consider in the present paper have been suggested by physics, namely by the classical field theory. We meet there functionals on the set of submanifolds of a fixed differentiable manifold $M$. In the standard canonical formulation states of the field are Cauchy's data for the field equations (i.e. submanifolds of some tensor bundles over space-time). Physical quantities (e.g., energy, momentum, etc.) are precisely functionals on the space of states. Since all interactions in physics propagate with finite speed (in view of hyperbolic character of field equations), physical quantities are functionals of local type, i.e., they have the following property: change of value caused by a local deformation of state does not depend on its shape outside the domain of deformation (precise difinition is given in section 2). As is known, functionals given by integrals of differential forms are of local type.

The following problem arises: are all functionals of local type given by integrals? In the present paper this problem is presented in all its generality. For sufficiently smooth functionals (belonging to the space $\mathfrak{F}_0$ whose precise definition is given in section 3) we have been able to show that the localness of a functional implies possibility of representing it by integrals of differential forms. Our results can be easily generalized for submanifolds with boundary, and for smooth functionals of lesser degree of smoothness ($\mathscr{F}_N$). In the latter case, the form which must be integrated is defined not in $M$ itself, but in a space enlarged by "higher order derivatives", i.e., in the bundle $\overset{k}{\bigwedge} T(M)$ or in one of its jet extensions.

These results lead to the possibility of construction of a local canonical formalism in the field theory. In such formalism physical quantities are *a priori* differential forms on multisymplectic manifolds and the operation of "Poisson bracket" is of finite-dimensional character (cf. the results of the author and K. Gawędzki which will be published elsewhere).

The set $\mathscr{P}_k$ of submanifolds which we consider here has no differentiable structure. Reasons for it are explained in the Appendix. In order to avoid this difficulty we introduce an apparently useful notion of the derivative of a functional along a complete vector field on $M$. The idea of such a derivative is evidently based on the deep concept of Ślebodziński from 1931 (cf. [4]), called by him the *Lie derivative*. It seems natural to call the introduced notion *Ślebodziński's derivative*.

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**2. Functionals of local type.** All constructions which we will make in this paper will be situated in a fixed $n$-dimensional, contractible, denumerable at infinity, simply connected at infinity, differentiable manifold $M$ of $C^\infty$ class. We will be concerned with the family $\mathscr{P}_k$ of all $k$-dimensional smooth submanifolds of $M$, which are without boundary, oriented, well imbedded and closed in $M$.

By *smoothness* we always mean that of $C^\infty$ class. For a given $\Omega \in \mathscr{P}_k$, a continuous homotopy

$$\Omega \times [0, 1] \ni (x, t) \to H(x, t) \in M, \qquad H(x, 0) \equiv x,$$

such that $\Omega_t = \{H(x, t): x \in \Omega\}$ belongs to $\mathscr{P}_k$ for all $t \in [0, 1]$ will be called a *regular homotopy* of $\Omega$. We will use the following notation: $H(\Omega) := \Omega_1$.

Definition. By the *support* of a homotopy $H$ we mean the set

$$\operatorname{supp} H := \{x \in \Omega: H(x, t) \neq x \text{ for some } t \in [0, 1]\}^-,$$

where the upper bar denotes the closure of the set.

Definition. A functional $F$ on $\mathscr{P}_k$ is said to be *of local type* if, for any regular homotopies $H_1$ and $H_2$ such that $\operatorname{supp} H_1 \cap \operatorname{supp} H_2 = \emptyset$, the following equality holds:

$$F\big(H_1(H_2(\Omega))\big) - F\big(H_2(\Omega)\big) = F\big(H_1(\Omega)\big) - F(\Omega). \tag{1}$$

One could say that change of value of $F$ caused by a local deformation $H_1$ does not depend on the shape of $\Omega$ outside the domain of this deformation.

In the field theory in physics we often meet the Bogoliubov condition

$$\frac{\delta}{\delta\Omega(x)} \frac{\delta}{\delta\Omega(y)} F = 0 \quad \text{for } x \neq y. \tag{2}$$

As will be shown, a precise mathematical meaning can be given to this condition and it is equivalent to (1) for some class of functionals.

Examples.

1° Let $\omega$ be a continuous differential $k$-form on $M$. If support of $\omega$ is compact, then

$$F(\Omega) := \int_{\Omega} \omega$$

is of local type, because both sides of (1) are equal to

$$\int_{H_1(\operatorname{supp} H_1)} \omega - \int_{\operatorname{supp} H_1} \omega,$$

independently of $H_2$.

2° Let $M$ be a Riemannian manifold. It is easy to see that the length of a curve, the area of a 2-dimensional surface, etc., are of local type. (Of course, such functionals are defined only on the set of compact submanifolds.)

**3. Smooth functionals on $\mathscr{P}_k$.** If we want to formulate conditions similar to the Bogoliubov condition (2), we must define in some way an operation of differentiation of functionals on $\mathscr{P}_k$. If we would consider only compact submanifolds, the differentiable structure in $\mathscr{P}_k$ could be defined, and it was done in [3]. But even such a structure is not useful for our purposes (cf. Appendix).

However, smoothness of functionals can be defined in the following way.

Let us denote by $\mathfrak{X}$ the set of all complete, smooth vector fields on $M$. Take $X \in \mathfrak{X}$. The field $X$ generates the group of diffeomorphisms $H_t^X: M \to M$, $t \in R$.

We use the notation $X(\Omega) := H_1^X(\Omega)$ (the image of $\Omega$ by $H_1^X$) and often, for convenience, we write $\Omega(X) := X(\Omega)$.

For any $\Omega \in \mathscr{P}_k$ we have a mapping

$$\mathfrak{X} \ni X \to \Omega(X) \in \mathscr{P}_k.$$

These mappings can be used to carry some structures from $\mathfrak{X}$ to $\mathscr{P}_k$, e.g., a topology.

Definition. By the *topology of compact convergence* of all derivatives in $\mathscr{P}_k$ we mean the topology induced by the family of mappings $\{X \to \Omega(X)\}_{\Omega \in \mathscr{P}_k}$ provided $\mathfrak{X}$ is equipped with the topology of compact convergence of all derivatives.

The last expression means that we take in $\mathfrak{X}$ the family of seminorms

$$|X|_{N,(\varkappa_1, O_1)\ldots(\varkappa_s, O_s)} = \sup_{|\alpha| \leqslant N} \sup_{i \leqslant s} \sup_{1 \leqslant j \leqslant n} \sup_{x \in O_i} |D^{\alpha}_{(i)} X^j_{(i)}(x)|,$$

where $a = (a_1, \ldots, a_n)$, $|a| = a_1 + \ldots + a_n$, $D^a$ is the Schwartz symbol

$$D^a_{(i)} f = \frac{\partial^{|a|} f}{\partial (x^1_{(i)})^{a_1} \ldots \partial (x^n_{(i)})^{a_n}},$$

$(\varkappa_i, O_i)$ is a coordinate chart in $M$ which can be extended onto some neighbourhood of $\bar{O}_i$, $O_i$ is pre-compact, and $X^j_{(i)}$ and $x^j_{(i)}$ are coordinates taken in the coordinate chart $(\varkappa_i, O_i)$.

Since mappings $X \to \Omega(X)$ are not injections, they cannot be treated as "coordinate charts" in $\mathscr{P}_k$, and so a differentiable structure cannot be carried from $\mathfrak{X}$ to $\mathscr{P}_k$. But we take the following

Definition. Let be given a field $X \in \mathfrak{X}$, a submanifold $\Omega \in \mathscr{P}_k$, and a functional $F$ defined on $\mathscr{P}_k$. By the *Ślebodziński derivative* of $F$ along $X$ we mean (if it does exist)

$$\nabla_X F(\Omega) := \lim_{t \to 0} \frac{1}{t} \Big( F\big(\Omega(tX)\big) - F(\Omega) \Big).$$

We say that $F$ is *of $C^1$ class* on $\mathscr{P}_k$ if $\nabla_X F(\Omega)$ does exist for all $\Omega \in \mathscr{P}_k$ and $X \in \mathfrak{X}$, and if it is linear and partially continuous in variable $X$ and partially continuous in variable $\Omega$.

By *linearity* we mean the property

$$\Big(X_i \in \mathfrak{X},\ a_i \in R,\ \sum a_i X_i \in \mathfrak{X}\Big) \Rightarrow \nabla_{\Sigma a_i X_i} F(\Omega) = \sum a_i \nabla_{X_i} F(\Omega)$$

(the space $\mathfrak{X}$ is not a vector space).

The following lemma is evidently true:

LEMMA 1. *If for each $x \in \Omega$ we have $X(x) \in T_x(\Omega)$ (the field $X$ is tangent to $\Omega$), then $\nabla_X F(\Omega) = 0$.*

For a fixed $X \in \mathfrak{X}$ expression $\nabla_X F(\Omega)$ defines a functional on $\mathscr{P}_k$. So, we can define the following "iterated" Ślebodziński's derivatives:

$$\nabla_{X_1} \ldots \nabla_{X_s} F(\Omega) := \nabla_{X_1}\Big(\nabla_{X_2} \ldots \big(\nabla_{X_s} F(\Omega) \ldots\big)\Big). \tag{3}$$

A functional $F$ is said to be *smooth* if each of its iterated Ślebodziński's derivatives do exist and is of $C^1$ class.

Remark. Iterated Ślebodziński's derivatives are not in any sense $n$-th derivatives of a function on any "model" vector space. They have not, therefore, the property of symmetry. Nevertheless, the following lemma is true:

LEMMA 2. *If $[X_1, X_2] = 0$, then $\nabla_{X_1} \nabla_{X_2} F(\Omega) = \nabla_{X_2} \nabla_{X_1} F(\Omega)$.*

Proof. If $X_1$ and $X_2$ commute, then so do the generated by them groups of diffeomorphisms. We define

$$R^2 \ni (t_1, t_2) \to \varphi(t_1, t_2) := F\Big(t_1 X_1\big(t_2 X_2(\Omega)\big)\Big) = F\Big(t_2 X_2\big(t_1 X_1(\Omega)\big)\Big).$$

To complete the proof we note that

$$\nabla_{X_1}\nabla_{X_2}F(\Omega) = \frac{\partial^2}{\partial t_1 \partial t_2}\varphi(0,0) = \frac{\partial^2}{\partial t_2 \partial t_1}\varphi(0,0) = \nabla_{X_2}\nabla_{X_1}F(\Omega).$$

COROLLARY. *Let be given an iterated Ślebodziński derivative with respect to the sequence* $(X_1, \ldots, X_s)$. *If any two neighbouring fields in this sequence commute, then they can be exchanged one for another.*

Definition. A functional $F$ is said to be *of* $C^1$ *class with respect to* 0*-norms* if its Ślebodziński's derivative is a continuous, linear operator on the space $\mathfrak{X}$ equipped with the topology of compact convergence (without derivatives), and if, for any fixed $X \in \mathfrak{X}$, it is continuous with respect to $\Omega$.

Our terminology derives from the fact that the topology in $\mathfrak{X}$ of compact convergence without derivatives is given by norms $|\ |_{0,(\varkappa_1,O_1)\ldots(\varkappa_s,O_s)}$ $(N = 0)$.

Of course, $\nabla_X F(\Omega)$ can be extended to a linear continuous form on the space of all smooth vector fields or (if $F$ is $C^1$ with respect to 0-norms) to the space of all continuous vector fields.

We will use the following useful criterion:

PROPOSITION. *If* $F$ *is of* $C^1$ *class with respect to* 0*-norms, then, for any* $X_i \in \mathfrak{X}$, *where* $i = 1, \ldots, s$, *the distribution*

$$\left(C_0^\infty(M)\right)^s \ni (f_1, \ldots, f_s) \to \nabla_{f_1X_1+\ldots+f_sX_s}F(\Omega)$$

*of* $s$ *variables is a finite measure (i.e., it is continuous with respect to the norm* $\|f\| = \sup|f(x)|$), *depending continuously on* $\Omega$.

Here and later on by a *measure* we mean not only a positive but an arbitrary regular distribution.

Proof. The uniform convergence $f_i^p \to f_i$ for $p \to \infty$ implies the almost uniform convergence of $f_1^pX_1 + \ldots + f_s^pX_s$.

It is not very difficult to show that the inverse is also true but we will not use this fact.

Definition. $F$ is *smooth with respect to* 0*-norms* if every its iterated Ślebodziński's derivative is of $C^1$ class with respect to 0-norms.

The set of all functionals smooth with respect to 0-norms will be denoted by $\mathfrak{F}_0(\mathscr{P}_k)$ or, simply, by $\mathfrak{F}_0$.

Similarly, if we take in $\mathfrak{X}$ the topology of compact convergence of derivatives of degree $p \leqslant N$, we can define spaces $\mathfrak{F}_N(\mathscr{P}_k) = \mathfrak{F}_N$. Elements of $\mathfrak{F}_N$ are said to be *smooth with respect to* $N$*-th norms*.

In the present paper we will consider only the space $\mathfrak{F}_0$. A generalization of our results to the case of $\mathfrak{F}_N$ is not very difficult.

Examples.

1° Let be given

$$F(\Omega) = \int_{\Omega} \omega,$$

where $\omega$ is a smooth $k$-form in $M$ whose support is compact. It can be easily shown (cf. [2]) that

$$\nabla_X F(\Omega) = \int_{\Omega} d\omega \llcorner X. \tag{4}$$

If we replace $\omega$ by $\omega_{p+1} := d\omega_p \llcorner X_{p+1}$ (where $\omega_0 = \omega$, $p = 1, 2 \ldots$), we see that all iterated Ślebodziński's derivatives are given by integrals similar to (4). Hence $F \in \mathfrak{F}_0$.

2° It is easy to see that the length of a curve and the area of a 2-dimensional surface are not smooth (even continuous) with respect to 0-norms, and one can easily show that they are smooth with respect to 1-norms.

**4. Representation theorem.** The space $\mathscr{P}_k$ is clearly not connected (cf. [3]). Its arcwise connected components are composed of submanifolds which can be joined one to another by a homotopy.

THEOREM. *Let $F \in \mathfrak{F}_0(\mathscr{P}_k)$ be a functional of local type.*

1° *If $k < n-1$, then, for every arcwise connected component $\mathscr{P}'_k \subset \mathscr{P}_k$, there exist a number $c \in R$ and a smooth differential $k$-form $\omega$ on $M$ such that, for every $\Omega \in \mathscr{P}'_k$, we have*

$$F(\Omega) = c + \int_{\Omega} \omega.$$

2° *If $\mathscr{P}'_k$ consists of non-compact submanifolds, then* $\operatorname{supp} \omega$ *is compact.*

3° *If $k = n-1$ and $\mathscr{P}'_k$ consists of compact submanifolds, then the thesis of* 1° *also holds true.*

4° *If $k = n-1$ and $\mathscr{P}'_k$ consists of non-compact submanifolds, then there exists a smooth $n$-form $a$ on $M$ such that* $\operatorname{supp} a$ *is compact and*

$$F(\Omega_1) - F(\Omega_2) = \int_{[\Omega_1, \Omega_2]} a,$$

*where by $[\Omega_1, \Omega_2]$ we denote the $n$-dimensional volume (possibly degenerated) between $\Omega_1$ and $\Omega_2$, oriented by the parametrisation $H^X(x, t)$ for any homotopy $H^X$ connecting $\Omega_1$ with $\Omega_2$.*

An idea of the proof may be sketched as follows.

Let be given $\Omega \in \mathscr{P}_k$, $X \in \mathfrak{X}$ and a $(k+1)$-dimensional volume $\Theta$ "swept" by $\Omega_t = \Omega(tX)$. If $\omega$ is a $k$-form, then $d\omega$ gives a smooth measure on $\Theta$

which will be denoted by $\mu_\Theta$. Now, if $\Omega^1, \Omega^2 \in \mathscr{P}_k$ lie in $\Theta$ (e.g., $\Omega^i = \Omega(f^i X)$, $f^i \in C^\infty(M)$), then, for the functional

$$F(\Omega) = c + \int_\Omega \omega,$$

we have

$$F(\Omega^1) - F(\Omega^2) = \int_{\Omega^1 - \Omega^2} \omega = \int_{[\Omega^1, \Omega^2]_\Theta} d\omega = \mu_\Theta([\Omega^1, \Omega^2]_\Theta) \tag{5}$$

(here we use the fact that $\partial \Omega^1 = \partial \Omega^2 = 0$, so $\partial[\Omega^1, \Omega^2]_\Theta = \Omega^1 - \Omega^2$). If we knew all measures $\mu_\Theta$, we would have the form $d\omega$. So, formula (5) will be the basis of our proof. It gives a possibility of defining measures $\mu_\Theta$ by $F$. Moreover, formula (5) states that $d\omega$ is uniquely determined by $F$. Of course, $\omega$ is not uniquely determined. For instance, the form $\omega + d\lambda$ is also good if $(k-1)$-form $\lambda$ has compact support.

**5. Proof of the representation theorem.** The following lemmas will be needed in the proof:

LEMMA 3. *Let $\mathscr{P}'_k$ be a component of $\mathscr{P}_k$ and let $\gamma \subset M$ be a $k$-dimensional, oriented, smooth submanifold imbedded in $M$. Then, for every $x \in \gamma$, there exist a neighbourhood $O_x$ of $x$ in $M$ and $\Omega_x \in \mathscr{P}'_k$ such that $\gamma \cap O_x = \Omega_x \cap O_x$ and orientations of both pieces are identical.*

Let $F$ be of $C^1$ class with respect to 0-norms. For every $X \in \mathfrak{X}$ and $\Omega \in \mathscr{P}_k$, the measure which gives a mapping $f \to \nabla_{fX} F(\Omega)$ will be denoted by $\nabla_X F(\Omega, x)\,dx$. Then

$$\nabla_{fX} F(\Omega) = \int_\Omega f(x) \nabla_X F(\Omega, x)\,dx$$

and

$$\nabla_X F(\Omega) = \int_\Omega \nabla_X F(\Omega, x)\,dx.$$

LEMMA 4. *If $F \in \mathfrak{F}_0$ is of local type, then, for every arcwise connected component $\mathscr{P}'_k \subset \mathscr{P}_k$ consisting of non-compact submanifolds, there exists a compact set $K \subset M$ such that all measures $\nabla_X F(\Omega, x)\,dx$ vanish beyond $K$.*

COROLLARY. *If assertions of lemma 4 are satisfied, then $\Omega_1 \cap K = \Omega_2 \cap K$ implies $F(\Omega_1) = F(\Omega_2)$ for any $\Omega_1, \Omega_2 \in \mathscr{P}'_k$.*

LEMMA 5. *If $X_1, \ldots, X_s \in \mathfrak{X}$ and $F$ is smooth, then*

$$(C_0^\infty(M))^s \ni (f_1, \ldots, f_s) \to \nabla_{f_1 X_1} \ldots \nabla_{f_s X_s} F(\Omega)$$

*is a distribution of $s$ variables, whose support is contained in $\Omega^s \subset M^s$.*

LEMMA 6. *A smooth functional $F$ is of local type if and only if, for every $X_1, X_2 \in \mathfrak{X}$ and $\Omega \in \mathscr{P}_k$, the support of the distribution*

$$(f_1, f_2) \to \nabla_{f_1 X_1} \nabla_{f_2 X_2} F(\Omega)$$

*is contained in the diagonal of $\Omega \times \Omega$, i.e., in $\{(x, x) : x \in \Omega\} \subset \Omega \times \Omega$.*

Lemma 6 gives the precise mathematical meanning to the Bogoliubov condition (2).

LEMMA 7. *If $F$ is smooth, $\Omega \in \mathscr{P}_k$ and $X_1, X_2 \in \mathfrak{X}$, then $X_1|\Omega = X_2|\Omega$ implies $\nabla_{X_1} F(\Omega) = \nabla_{X_2} F(\Omega)$.*

The following Lemma is fundamental in our proof:

LEMMA 8. *Let $X \in \mathfrak{X}$ and $F \in \mathfrak{F}_0$ be such that all measures $\nabla_Y F(\Omega, y)dy$ have compact supports (it means that either $\Omega$ is compact or all these measures vanish beyond some compact $K \subset M$). Let $X$ be transversal to $\Omega$ (i.e., $X(x) \notin T_x(\Omega)$).*

*If $F$ is of local type, then*

$$\nabla_{\varphi X} \nabla_{fX} F(\Omega) = \int_{\Omega} \varphi(x) \frac{d}{dt} \{H^X_{-t}(\nabla_{fX} F(\Omega_t, x)dx\}|_{t=0},$$

*where $\Omega_t = \Omega(tX)$ and $H^X_\tau(\mu)$ is a measure carried from $\Omega_t$ to $\Omega_{t+\tau}$ by $H^X_t$.*

LEMMA 9. *Let $\mu$ be any measure on an $m$-manifold $\Theta$. If for any $X \in \mathfrak{X}$ the Lie derivative $(\mathfrak{L}_X)^s \mu$ is also a measure, then $\mu$ is a smooth differential $m$-form on $\Theta$.*

Now we are ready to pass to the proof of the theorem. Take any simple $(k+1)$-vector $w \in \overset{k+1}{\bigwedge}_s T_{x_0}(M)$ and choose a coordinate chart in a neighbourhood of $x_0$ such that $w$ is represented as

$$w = \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^{k+1}}.$$

Using lemma 3, we can also choose $\Omega \in \mathscr{P}'_k$ such that, for some neighbourhood $O \subset M$ of $x_0$, the manifold

$$\{(x^1, \ldots, x^n): x^{k+1} = x^{k+2} = \ldots = x^n = 0\}$$

(taken with its natural orientation induced by coordinates $(x^1, \ldots, x^k)$) coincides with $\Omega$. Let us take any field $X$ for which $X|O = \partial/\partial x^{k+1}$ (such an $X$ obviously exists). Define also $(k+1)$-dimensional oriented manifold

$$\Theta = \{(x^1, \ldots, x^n): x^{k+2} = \ldots = x^n = 0\} \cap O$$

with the orientation given by $(x^1, \ldots, x^{k+1})$.

Now we will try to define the measure $\mu$ on $\Theta$. Suppose $f$ is a smooth function on $\Theta$, whose support is compact. Choose any $\psi \in C^\infty(M)$ such that $\psi|\Theta = f$. Set

$$\mu(f) := \int dt \nabla_{\psi X} F(\Omega_t) = \int dt \int_{\Omega_t} \psi(x) \nabla_X F(\Omega_t, x)dx, \tag{6}$$

where $\Omega_t = \Omega(tX)$.

Boundaries of the integral $\int dt$ are chosen in a way to cover whole support of $f$.

Since left-hand side of (6) does not depend on a particular choice of $\psi$ (lemma 7), we simply write

$$\mu(f) = \int dt\, \nabla_{fX} F(\Omega_t).$$

Now we must show that $\mu$ defined in this way does not depend on a particular choice of a coordinate chart $(x^1, \ldots, x^{k+1})$ in $\Theta$. Assume that the congruence $\Omega_t$ and the field $X$ are deformed by a homotopy $H_\varepsilon^Y$ ($\varepsilon \in [0, 1]$), where $Y \in \mathfrak{X}$. Denote

$$\Omega_t^\varepsilon = H_\varepsilon^Y(\Omega_t), \qquad X^\varepsilon = H_\varepsilon^Y(X), \qquad \text{and} \qquad \mu_\varepsilon(f) := \int dt\, \nabla_{fX^\varepsilon} F(\Omega_t^\varepsilon).$$

Now,

$$\begin{aligned}(7)\qquad \frac{d}{d\varepsilon}\mu_\varepsilon(f)(0) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int dt\big(\nabla_{fX^\varepsilon}F(\Omega_t^\varepsilon) - \nabla_{fX}F(\Omega_t)\big)\\ &= \int dt \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{\nabla_{fX^\varepsilon}F(\Omega_t^\varepsilon) - \nabla_{fX}F(\Omega_t^\varepsilon)\} + \\ &\qquad + \int dt \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{\nabla_{fX}F(\Omega_t^\varepsilon) - \nabla_{fX}F(\Omega_t)\}.\end{aligned}$$

The first term of (7) is equal to

$$(8)\qquad \lim_{\varepsilon\to 0} \nabla_{\varepsilon^{-1}f(X^\varepsilon - X)}F(\Omega_t^\varepsilon) = \nabla_{f\lim \varepsilon^{-1}(X^\varepsilon - X)}F(\Omega_t).$$

The last equality can be explained as follows. The mapping

$$(9)\qquad \mathfrak{X}\times R \ni (Z, \varepsilon) \to \nabla_Z F(\Omega_t^\varepsilon) \in R$$

is partially continuous in both variables and linear in the first variable. But the value of $\nabla_Z F(\Omega_t^\varepsilon)$ does not depend on the value of $Z$ beyond some compact set $D$ ($D$ is either a neighbourhood of $K$ from lemma 4 or a neighbourhood of the set $\Omega \subset M$ if $\Omega$ is compact). So, we can project this mapping to the quotient space $\mathfrak{X}' = \mathfrak{X}/$(fields vanishing on $D$).

Since $\mathfrak{X}'$ is a Fréchet space, we can use the generalized Mazur-Orlicz theorem (cf. [1]) which states that (9) is continuous. This fact implies (8). But

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(X^\varepsilon - X) = [Y, X].$$

Since the field $Y$ is tangent to $\Theta$, it can be decomposed into two components: the first — tangent to the congruence $\Omega_t$ and the second — parallel to $X$,

$$Y(x) = Y_{\parallel}(x) + \varphi(x)\cdot X(x).$$

Therefore

$$[Y, X] = [Y_{\|}, X] + [\varphi X, X] = [Y_{\|}, X] + \frac{d\varphi}{dt} X.$$

But the field $Z = [Y_{\|}, X]$ is tangent to the congruence $\Omega_t$ ($Y_{\|}$ is tangent and $X$ leaves the congruence invariant), and so $\nabla_{fZ} F(\Omega_t) = 0$. Finally, the first term of (7) is equal to

$$\int dt \nabla_{(d\varphi/dt)fX} F(\Omega_t).$$

Let us compute the second term of (7). We have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \{\nabla_{fX} F(\Omega_t^\varepsilon) - \nabla_{fX} F(\Omega_t)\} = \nabla_Y \nabla_{fX} F(\Omega_t)$$
$$= \nabla_{Y_{\|}} \nabla_{fX} F(\Omega_t) + \nabla_{\varphi X} \nabla_{fX} F(\Omega_t) = \nabla_{\varphi X} \nabla_{fX} F(\Omega_t).$$

Now we approach the most important part of the proof. Using lemma 8, we can write

$$\nabla_{\varphi X} \nabla_{fX} F(\Omega_t) = \int_{\Omega_t} \varphi(x) \frac{d}{d\tau} \{H_{-\tau}^X(\nabla_{fX} F(\Omega_{t+\tau}, x) dx)\}\big|_{\tau=0}. \tag{10}$$

For any $x = (x^1, \ldots, x^k, x^{k+1})$ let us denote $p = (x^1, \ldots, x^k)$ and $t = x^{k+1}$. So, we write $x = (p, t)$ and $x = H_t^X(p)$.

Defining a family of measures on $\Omega$ by

$$\nu_t(p) dp = \nu(p, t) dp := H_{-t}^X\{\nabla_{fX} F(\Omega_t, x) dx\},$$

we can write

$$\int dt \nabla_{(d\varphi/dt)fX} F(\Omega_t) = \int dt \int_{\Omega_t} \frac{d\varphi}{dt}(x) \nabla_{fX} F(\Omega_t, x) dx \tag{11}$$
$$= \int dt \int dp \frac{d\varphi}{dt}(p, t) \nu(p, t).$$

Similarly, using (10),

$$\int dt \nabla_{\varphi X} \nabla_{fX} F(\Omega_t) = \int dt \int_{\Omega} \varphi(p, t) \frac{d}{dt} \nu(p, t) \tag{12}$$

(expression $d\nu(p, t)/dt$ is to be understood in the sense of the theory of distributions).

If we gather (11) and (12), we finally obtain

$$\frac{d}{d\varepsilon} \mu_\varepsilon(f)(0) = \int\int \left(\frac{d\varphi}{dt} \nu + \varphi \frac{d\nu}{dt}\right) dt\, dp = 0.$$

Similarly, we can obtain $d[\mu_\varepsilon(f)(\varepsilon_0)]/d\varepsilon = 0$ for every $\varepsilon_0$. In order to see this, one must put $\varepsilon_0$ instead of 0 and $\varepsilon' = \varepsilon - \varepsilon_0$ instead of $\varepsilon$. So, $\mu_\varepsilon = \mu$ for every $\varepsilon$.

But since (locally) every two congruences $\{\Omega_t\}$ lying in the same component $\mathscr{P}'_k$ can be connected by a homotopy $H^Y$, $\mu$ is well defined and does not depend on a particular choice of coordinates in $\Theta$.

Now we will show that $\mu$ is a smooth $(k+1)$-form on $\Theta$. We have

$$(13) \qquad \left(\frac{\partial}{\partial x^{k+1}}\mu\right)(f) = \int dt \int_{\Omega_t} f(x) \frac{d}{d\tau}\{H^X_{-\tau}(\nabla_X F(\Omega_{t+\tau}, x)\,dx)\}\big|_{\tau=0}$$
$$= \int dt\, \nabla_{fX}\nabla_X F(\Omega_t),$$

the last equality being again a consequence of lemma 8. But $F \in \mathfrak{F}_0$, and so (13) implies that $\partial\mu/\partial x^{k+1}$ is a measure. By iteration we obtain

$$\left(\frac{\partial^s}{\partial(x^{k+1})^s}\mu\right)(f) = \int dt\,\nabla_{fX}(\nabla_X)^s F(\Omega_t),$$

thus again a measure. But coordinate chart can be arbitrarily chosen, and so we can conclude that the Lie derivative $(\pounds_X)^s\mu$ is a measure for every $X \in \mathfrak{X}$, $s = 1, 2 \ldots$ Using lemma 9, we conclude that $\mu$ is a smooth differential $(k+1)$-form on $\Theta$. It will be denoted by $a_\Theta$. Let us compute how it acts on our $(k+1)$-vector $w$. In order to do this, we take a $\delta$-sequence of functions $\varphi_s \in C_0^\infty(R)$ (the space of smooth functions with compact supports), $\int\varphi_s(t)\,dt = 1$ for every $s$, $\operatorname{supp}\varphi_s \to \{0\}$. Using any coordinate chart on $\Theta$ in a neighbourhood of $x_0$, we can define

$$\psi_s(x) := \prod_{i=1}^{k} \varphi_s(x^i - x_0^i),$$

where $x_0^i$ are coordinates of $x_0$. Then

$$(14) \qquad \langle a_\Theta(x_0), w\rangle = \lim_{s\to\infty}\int \varphi_s(t)\,dt \int_{\Omega_t} \psi_s(x)\,\nabla_X F(\Omega_t, x)\,dt$$
$$= \lim_{s\to\infty} \nabla_{\psi_s X} F(\Omega).$$

These considerations prove also that $\nabla_X F(\Omega, x)\,dx$ is a smooth $k$-form on $\Omega$ and that

$$\int_\Omega f(x)\,\nabla_X F(\Omega, x)\,dx = \int_\Omega a_\Theta \llcorner f X.$$

Finally,

$$(15) \qquad \nabla_X F(\Omega, x)\,dx = a_\Theta \llcorner X.$$

Similar construction can be made for every $w \in \overset{k+1}{\bigwedge}_s T(M)$ at every point $x_0 \in M$. But we must prove that the result $\langle a_\Theta(x_0), w\rangle$ does not depend on a particular choice of $\Theta$, i.e., that it gives a function on the space $\overset{k+1}{\bigwedge}_s T(M)$.

Let $\Theta_1$ and $\Theta_2$ have the same tangent element $w$ at $x_0$. Assume first that $\Theta_1$ and $\Theta_2$ are tangent to each other along a whole $k$-dimensional submanifold which we denote by $\Omega$. In such a case we can choose $X_1, X_2 \in \mathfrak{X}$ such that $\Theta_1$ and $\Theta_2$ are "swept" by $\Omega(tX_1)$ and $\Omega(tX_2)$ (recall that all these considerations have only local character). Moreover, $X_1$ and $X_2$ can be chosen in such a manner that they will be tangent to one another on $\Omega$, i.e., $X_1|\Omega = X_2|\Omega$. Using lemma 7 and formula (15), we see that

$$\langle a_{\Theta_1} \llcorner X_1, \tilde{w}\rangle = \langle a_{\Theta_2} \llcorner X_2, \tilde{w}\rangle,$$

where $w$ is a $k$-vector tangent to $\Omega$ at $x_0$ such that

$$\tilde{w} \wedge X_1(x_0) = \tilde{w} \wedge X_2(x_0) = w.$$

Hence

$$\langle a_{\Theta_1}(x_0), w\rangle = \langle a_{\Theta_2}(x_0), w\rangle. \tag{16}$$

In the general case, if $\Theta$ and $\Theta'$ are tangent one to another only at $x_0$, we can build a sequence of manifolds $\Theta_0, \Theta_1, \ldots, \Theta_{k+1}$, where $\Theta_0 = \Theta$ and $\Theta_{k+1} = \Theta'$, such that every two neighbouring manifolds $\Theta_i$ and $\Theta_{i+1}$ are tangent along a whole $k$-dimensional submanifold passing by $x_0$. If we use (16) $k+1$ times, we obtain $a_\Theta(x_0) = a_{\Theta'}(x_0)$. So, we can drop the index $\Theta$ in $a_\Theta$ and write simply $a$.

Our $a$ is now a function on the bundle of simple $k$-vectors in $M$. Its linearity is an immediate consequence of (14) or (15) and of linearity of Ślebodziński's derivative. Thus, $a$ can be extended to the whole $\overset{k+1}{\bigwedge} T(M)$ as a differential $(k+1)$-form.

Finally,

$$\int_\Omega a \llcorner X = \nabla_X F(\Omega), \qquad a \llcorner X = \nabla_X F(\Omega, x)\, dx.$$

Now we can use the formula (4), where we replace $\omega$ by $a \llcorner X$ and $F$ by $\nabla_X F$. We obtain

$$\nabla_{fY} \nabla_X F(\Omega) = \int_\Omega d(a \llcorner X) \llcorner fY = \int_\Omega \pounds_{fY}(a \llcorner X)$$

and, similarly, for all iterated Ślebodziński's derivatives.

Taking any coordinate chart $(\varkappa, O)$ and

$$\Omega(x^{k+1}, \ldots, x^n) = \{(x^1, \ldots, x^n) : x^{k+1} = \text{const}, \ldots, x^n = \text{const}\},$$

we can build a measure in $M \cap O$,

$$\lambda = a \wedge dx^{k+2} \wedge \ldots \wedge dx^n = a_{1,\ldots,k+1} dx^1 \wedge \ldots \wedge dx^n,$$

where

$$a = \sum a_{i_1,\ldots,i_{k+1}} dx^{i_1} \wedge \ldots \wedge dx^{i_{k+1}}.$$

Denote $X_i = \partial/\partial x^i$,

$$\lambda(f) = \int dx^{k+1} \wedge \ldots \wedge dx^n \int\limits_{\Omega(x^{k+1},\ldots,x^n)} a \llcorner X_{k+1} \cdot f$$
$$= \int dx^{k+1} \wedge \ldots \wedge dx^n \nabla_{fX_{k+1}} F\big(\Omega(x^{k+1}, \ldots, x^n)\big)$$

(the last formula proves that $\lambda$ is indeed a measure). For $i \geqslant k+1$ we have

$$\left(\frac{\partial}{\partial x^i}\lambda\right)(f) = \int dx^{k+1} \wedge \ldots \wedge dx^n \int\limits_{\Omega(x^{k+1},\ldots,x^n)} f(x)\; £_{X_i}(a_{1,\ldots,k+1} dx^1 \wedge \ldots \wedge dx^k)$$
$$= \int dx^{k+1} \wedge \ldots \wedge dx^n \int\limits_{\Omega(x^{k+1},\ldots,x^n)} f(x)\{d(a_{1,\ldots,k+1}) \wedge dx^1 \wedge \ldots \wedge dx^k \llcorner X_i -$$
$$-d(a_{1,\ldots,k+1} dx^1 \wedge \ldots \wedge dx^k \llcorner X_i)\}$$
$$= \int dx^{k+1} \wedge \ldots \wedge dx^n \int\limits_{\Omega(x^{k+1},\ldots,x^n)} d(a_{1,\ldots,k+1}) \wedge dx^1 \wedge \ldots \wedge dx^k \llcorner fX_i$$
$$= \int dx^{k+1} \wedge \ldots \wedge dx^n \int\limits_{\Omega(x^{k+1},\ldots,x^n)} £_{fX_i}(a \llcorner X_{k+1})$$
$$= \int dx^{k+1} \wedge \ldots \wedge dx^n \nabla_{fX_i} \nabla_{X_{k+1}} F\big(\Omega(x^{k+1}, \ldots, x)\big).$$

Thus $\partial\lambda/\partial x^i$ is again a measure. For $i \leqslant k$ we deduce the same result from the fact that $a|\Theta$ is smooth for every $\Theta$. Iterating this procedure, we see (lemma 9) that $\lambda$ is a smooth differential $n$-form. It means that $a_{1,\ldots,k+1}$ is smooth. Similarly, we can prove that every $a_{i_1,\ldots,i_{k+1}}$ is smooth.

It means that $a$ is smooth (we have shown previously that $a|\Theta$ is smooth for every $\Theta$).

Now let be given two manifolds $\Omega, \Omega_1 \in \mathscr{P}'_k$ connected by the field $X \in \mathfrak{X}$. We have

$$F(\Omega_1) - F(\Omega) = \int dt \nabla_X F(\Omega_t) = \int dt \int\limits_{\Omega_t} a \llcorner X = \int\limits_{[\Omega, \Omega_1]_X} a,$$

where by $[\Omega, \Omega_1]_X$ we denote the $(k+1)$-dimensional volume (possibly degenerated) "swept" by the congruence $\Omega_t$, $t \in [0, 1]$.

Hence we have proved part 4° of our theorem.

To show the rest we must prove that $da = 0$. If we knew this, then, taking a $k$-form for which $d\omega = a$ ($M$ is contractible), we would have

$$\int_{[\Omega, \Omega_1]X} d\omega = \int_{\partial[\Omega, \Omega_1]X} \omega = \int_{\Omega_1} \omega - \int_{\Omega} \omega,$$

because of $\partial\Omega = \partial\Omega_1 = 0$. Choosing any fixed $\Omega_0$ and putting

$$c := F(\Omega_0) - \int_{\Omega_0} \omega,$$

we would have, in the case of compact submanifolds,

$$F(\Omega) = c + \int_{\Omega} \omega.$$

Thus we would have proved parts 1° and 3°. In the non-compact case this construction can also be made provided we can choose $\omega$ whose support is compact. But $\operatorname{supp} a$ is compact and $k+1 < n$, so in a contractible and simply connected at infinity manifold $M$ one can choose a $k$-form $\omega$ with a compact support.

The only thing which remains to prove is the equality $da = 0$. Take any coordinate chart $(\varkappa, O)$ in a neighbourhood of $x_0$. Let $\Omega$ be such that

$$\Omega \cap O = \{(x^1, \ldots x^n): x^{k+1} = \ldots = x^n = 0\}.$$

Let the field $X$ be given in $O$ by the formula

$$X(x) = \begin{cases} (0, \ldots, 0) & \text{if } (x^1)^2 + \ldots + (x^k)^2 > r^2, \\ (0, \ldots, 0, \sqrt{r^2 - (x^1)^2 - \ldots - (x^k)^2}, 0, \ldots, 0) & \text{elsewhere}, \end{cases}$$

where the only non-vanishing component of $X$ is $X^{k+1}$.

Let $Y$ be given by a similar formula, but the only non-vanishing component let be $Y^{k+2}$. Take the third field

$$Z(x) = (\underbrace{0, \ldots, 0}_{k \text{ times}}, -x^{k+2}, x^{k+1}, 0, \ldots, 0)$$

and the $(k+2)$-sphere

$$C = \{(x^1, \ldots, x^n): (x^1)^2 + \ldots + (x^{k+2})^2 \leqslant r^2, x^{k+3} = \ldots = x^n = 0\}.$$

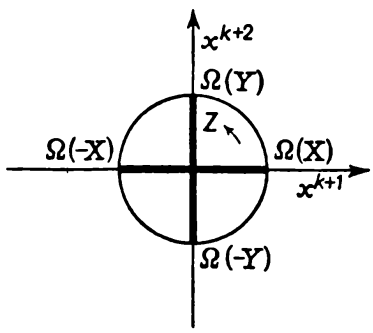


Fig. 1

All this may be projected parallely to $\Omega$ into the plane $(x^{k+1}, x^{k+2})$ (see Fig. 1).

We see that

$$\int_{\partial C} a = \int_{[\Omega(X),\Omega(Y)]Z} a + \int_{[\Omega(Y),\Omega(-X)]Z} a + \int_{[\Omega(-X),\Omega(-Y)]Z} a + \int_{[\Omega(-Y),\Omega(X)]Z} a.$$

If fields $X$ and $Y$ were smooth, we could write

$$\int_{\partial C} a = F(\Omega(Y)) - F(\Omega(X)) + F(\Omega(-X)) - F(\Omega(Y)) +$$
$$+F(\Omega(-Y)) - F(\Omega(-X)) + F(\Omega(X)) - F(\Omega(-Y)) = 0.$$

But fields $X$ and $Y$ can be approximated by smooth fields. Thus the formula

$$\int_{\partial C} a = 0$$

still holds. Hence

$$\int_C da = \int_{\partial C} a = 0$$

for every "sphere" $C$. It means that $da = 0$.

**6. Proofs of lemmas.**

Proof of lemma 3. Let us take any $\Omega \in \mathscr{P}'_k$. If $x \notin \Omega$, then there exists a field $X \in \mathfrak{X}$ such that $x \in \Omega_1 = \Omega(X)$. In order to show this, join point $x$ with an arbitrary $x_0 \in \Omega$ by a smooth curve. Let $x_1$ be the last point of this curve lying on $\Omega$. Take a smooth vector field $\tilde{X}$ on this curve such that $H_1^{\tilde{X}}(x_1) = x$. The field $\tilde{X}$ can be extended to a complete field $X \in \mathfrak{X}$. We see that $x \in \Omega(X)$.

Now let $(\varkappa, O_1)$ be a coordinate chart in a neighbourhood of $x$ such that $\Omega_1 \cap \bar{O}_1$ and $\gamma \cap \bar{O}_1$ are both $k$-dimensional discs. But there exists a homotopy which joins them, and which does not move the point $x$. Let $Y$ be a field in $O_1$ realizing this homotopy. Choose a smooth function $g$ on $M$, whose support lies in $O_1$ and which is equal to unity in some neighbourhood $O$ of $x$. Then $\Omega_x := \Omega_1(gY)$ has the needed property.

Proof of lemma 4. We have $M = \bigcup_{i=1}^{\infty} K_i$, $K_i$ compact. Let $\tilde{K}_1$ be compact such that $K_1 \subset \tilde{K}_1$ and $M - \tilde{K}_1$ is connected. If $\tilde{K}_1$ has not the demanded property, then take any $\Omega \in \mathscr{P}'_k$, $\tilde{X}_1 \in \mathfrak{X}$ and $x_1 \notin \tilde{K}_1$ such that $\nabla_{\tilde{X}_1} F(\Omega, x_1) dx \neq 0$. So, we can choose a field $X_1$ whose support is compact and such that $\nabla_{X_1} F(\Omega) = 1$. Take $\tilde{K}_2$ such that $\operatorname{int} \tilde{K}_2 \supset \{K_2 \cup \operatorname{supp} X_1\}$ and that $M - \tilde{K}_2$ is connected. If $\tilde{K}_2$ has not the demanded property, then take $x_2 \notin \tilde{K}_2$, $\tilde{X}_2 \in \mathfrak{X}$ and $\Omega_2 \in \mathscr{P}'_k$ such that $\nabla_{\tilde{X}_2} F(\Omega_2, x_2) \neq 0$. Using the method given in the proof of lemma 3, we can choose the field $Y_2 \in \mathfrak{X}$ such that $\operatorname{supp} Y_2 \subset M - \tilde{K}_2$ is compact and that $\Omega(Y_2) = \Omega_2$ in a neighbourhood of $x_2$. So, we can choose a field

$X_2 \in \mathfrak{X}$ whose compact support belongs to $M-\tilde{K}_2$ and for which $\nabla_{X_2} F(\Omega(Y_2)) = 1$. Similarly, we can build by induction the sequence of compact $\tilde{K}_i$ (such that $\operatorname{int}\tilde{K}_i \supset K_i$), $X_i$, $Y_i$ (whose compact supports belong to $M-\tilde{K}_i$ and to $\operatorname{int}\tilde{K}_{i+1}$) such that $\nabla_{X_i} F(\Omega(Y_i)) = 1$.

Let us define two complete vector fields by

$$X = \sum_{i=1}^{\infty} X_i \quad \text{and} \quad Y = \sum_{i=1}^{\infty} Y_i.$$

Both sums exist, because all summands have disjoint supports. Denote $\Omega^0 = \Omega(Y)$. Since $F$ is of local type, we have

$$\nabla_{X_i} F(\Omega^0) = \nabla_{X_i} F(\Omega(Y_i)) = 1.$$

Thus $\nabla_X F(\Omega^0)$ does not exist ($\nabla_X F(\Omega, x)dx$ is not finite) which contradicts our assertions.

Lemma 5 is obvious. If any $f_i$ has a support beyond $\Omega$, then diffeomorphisms $\{H_t^{f_i X_i}\}$ do not move $\Omega$ and the Ślebodziński derivative vanishes.

Proof of lemma 6. If $\operatorname{supp} f_1 \cap \operatorname{supp} f_2 = \emptyset$, then $\operatorname{supp} H^{f_1 X_1} \cap \cap \operatorname{supp} H^{f_2 X_2} = \emptyset$ and $H^{f_1 X_1}$, $H^{f_2 X_2}$ commute. We define

$$\varphi(t_1, t_2) := F\big(H_{t_1}^{f_1 X_1}(H_{t_2}^{f_2 X_2}(\Omega))\big) = F\big(H_{t_2}^{f_2 X_2}(H_{t_1}^{f_1 X_1}(\Omega))\big).$$

But $F$ is of local type, and so

$$\varphi(t_1, t_2) - \varphi(0, t_2) = \varphi(t_1, 0) - \varphi(0, 0),$$

which implies

$$\nabla_{f_1 X_1} \nabla_{f_2 X_2} F(\Omega) = \frac{\partial^2}{\partial t_1 \partial t_2} \varphi(0, 0) = 0.$$

The inverse is obvious.

Proof of lemma 7. It suffices to prove that $\nabla_X F(\Omega) = 0$ when $X | \Omega = 0$. But it is an immediate consequence of lemma 1.

Proof of lemma 8. Using partition of unity, we can reduce our problem to the case where $K$ is contained in a domain of a coordinate chart $(\varkappa, O)$. Let $(\varkappa, O)$ be such that $(x^1, \ldots, x^k)$ gives a coordinate chart on $\Omega$. Assume at first that $\varphi$ and $f$ are constant on integral curves of the field $X$. In this case $[\varphi X, fX] = 0$ and

$$S(\varphi, f) := \nabla_{\varphi X} \nabla_{fX} F(\Omega) = \nabla_{fX} \nabla_{\varphi X} F(\Omega)$$

is a bilinear, partially continuous functional on the space $C(\Omega \cap D) \times \times C(\Omega \cap D)$, where $D$ is an open, pre-compact neighbourhood of $K$. We divide the whole $O$ into cubes

$$A_a^\varepsilon := \{(x^1, \ldots, x^k) : \varepsilon a^i \leqslant x^i \leqslant \varepsilon(a^i + 1)\},$$

where $a = (a^1, \ldots, a^k)$ is a multi-index ($a^i$ are integers). Take any sequence $f_i \in C(\Omega \cap D)$, $0 \leqslant f_i \leqslant 1$, such that

$$f_i \xrightarrow[i\to\infty]{} 1_{A_a^\varepsilon} \text{ pointwise,}$$

where

$$1_A(x) := \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

If $S(\varphi, \cdot)$ is a measure on $\Omega \cap D$, then using the Lebesgue theorem we have, for any $\varphi \in C(\Omega \cap D)$,

$$S(\varphi, f_j) \to S(\varphi, 1_{A_a^\varepsilon}).$$

This means that $S(\cdot, f_j) \to S(\cdot, 1_{A_a^\varepsilon})$ in the sense of the simple topology in $C(\Omega \cap D)'$. Using the Banach-Steinhaus theorem, we see that $S(\cdot, 1_{A_a^\varepsilon})$ is again an element of $C(\Omega \cap D)'$, so it is a Borel-measure on $\Omega \cap D$. We will show that, for any $\varphi \in C(\Omega \cap D)$,

$$S(1_{A_a^\varepsilon}, \varphi \cdot 1_{A_\beta^\varepsilon}) = 0 \quad \text{for } a \neq \beta.$$

It is not very difficult to see that there exist sequences $f_i, g_j \in C(\Omega \cap D)$, whose values lie in $[0, 1]$, which satisfy

$$f_i \to 1_{A_a^\varepsilon} \quad \text{and} \quad g_j \to 1_{A_\beta^\varepsilon},$$

and for which at least one of two statements

(17) $$\operatorname{supp} f_i \cap \operatorname{supp} g_j = \emptyset \quad \text{for } i \leqslant j$$

or

(18) $$\operatorname{supp} f_i \cap \operatorname{supp} g_j = \emptyset \quad \text{for } i \geqslant j$$

is true (e.g., if $\beta = (a^1, \ldots, a^{s-1}, a^s+1, a^{s+1}, \ldots, a^k)$, then (17) is true).

If, e.g., (17) is true, then, using twice the Lebesque theorem and lemma 6, we obtain

$$0 \underset{i\leqslant j}{\equiv} S(f_i, \varphi \cdot g_j) \xrightarrow[j\to\infty]{} S(f_i, \varphi \cdot 1_{A_\beta^\varepsilon}) \xrightarrow[i\to\infty]{} S(1_{A_a^\varepsilon}, \varphi \cdot 1_{A_\beta^\varepsilon}).$$

If (18) is true, we must change the order of computing the limits. Denote now

$$x_a^\varepsilon := (\varepsilon a^1, \ldots, \varepsilon a^k) \in \Omega \cap D, \qquad \varphi_\varepsilon := \sum_a \varphi(x_a^\varepsilon) \cdot 1_{A_a^\varepsilon}.$$

Of course, $\varphi_\varepsilon \xrightarrow[\varepsilon\to 0]{} \varphi$ pointwise. Thus, using again the Lebesgue theorem, we obtain

$$S(\varphi, f) = \lim_{\varepsilon\to 0} S(\varphi_\varepsilon, f) = \lim_{\varepsilon\to 0} \sum_a S\big(1_{A_a^\varepsilon}, \varphi(x_a^\varepsilon) \cdot f\big).$$

But

$$S(1_{A^\varepsilon_\alpha}, \varphi(x^\varepsilon_\alpha)\cdot f) = S\big(1_{A^\varepsilon_\alpha}, \varphi(x^\varepsilon_\alpha)\sum_\beta 1_{A^\varepsilon_\beta}\cdot f\big)$$

$$= \sum_\beta S\big(1_{A^\varepsilon_\alpha}, \varphi(x^\varepsilon_\alpha)1_{A^\varepsilon_\beta}\cdot f\big) = S\big(1_{A^\varepsilon_\alpha}, \varphi(x^\varepsilon_\alpha)1_{A^\varepsilon_\alpha}\cdot f\big)$$

$$= \sum_\beta S\big(1_{A^\varepsilon_\alpha}, \varphi(x^\varepsilon_\beta)1_{A^\varepsilon_\beta}\cdot f\big) = S(1_{A^\varepsilon_\alpha}, \varphi_\varepsilon\cdot f).$$

Therefore

$$S(\varphi, f) = \lim_{\varepsilon\to 0}\sum_\alpha S(1_{A^\varepsilon_\alpha}, \varphi_\varepsilon\cdot f) = \lim_{\varepsilon\to 0} S(1, \varphi_\varepsilon\cdot f) = S(1, \varphi\cdot f).$$

It means that

$$\nabla_{\varphi X}\nabla_{fX}F(\Omega) = \nabla_X\nabla_{\varphi f X}F(\Omega) = \frac{d}{dt}\int_{\Omega_t}\varphi(x)\nabla_{fX}F(\Omega_t, x)\,dx\Big|_{t=0}$$

$$= \int_\Omega \varphi(x)\frac{d}{dt}\{H^X_{-t}(\nabla_{fX}F(\Omega_t, x)\,dx)\}\Big|_{t=0}$$

because $\varphi$ is constant on integral curves of $X$.

In the general case, denote by $\tilde{f}$ and $\tilde{\varphi}$ the functions which coincide with $f$ and $\varphi$ on $\Omega\cap D$ and which are constant on integral curves of $X$. Using lemma 7, we have

$$\nabla_{\varphi X}\nabla_{fX}F(\Omega) = \nabla_{\tilde{\varphi}X}\nabla_{fX}F(\Omega) = \lim_{t\to 0}\frac{1}{t}\{\nabla_{fX}F(\Omega(t\tilde{\varphi}X)) - \nabla_{fX}F(\Omega)\}$$

$$= \lim_{t\to 0}\frac{1}{t}\{\nabla_{fX}F(\Omega(t\tilde{\varphi}X)) - \nabla_{\tilde{f}X}F(\Omega(t\tilde{\varphi}X)) + \nabla_{\tilde{f}X}F(\Omega(t\tilde{\varphi}X)) - \nabla_{fX}F(\Omega)\}$$

$$= \lim_{t\to 0}\nabla_{t^{-1}(f-\tilde{f})X}F(\Omega(t\tilde{\varphi}X)) + \nabla_{\tilde{\varphi}X}\nabla_{\tilde{f}X}F(\Omega)$$

$$= \nabla_{\varphi\cdot X(f)\cdot X}F(\Omega) + \nabla_X\nabla_{\tilde{\varphi}\tilde{f}X}F(\Omega).$$

On the other hand,

$$\int_\Omega \varphi(x)\frac{d}{dt}\{H^X_{-t}(\nabla_{fX}F(\Omega_t, x)\,dx)\}|_{t=0}$$

$$= \frac{d}{dt}\int_\Omega \tilde{\varphi}(p)f(p, t)\nabla_X F(\Omega_t, p)\,dp|_{t=0}$$

$$= \int_\Omega \tilde{\varphi}(p)\frac{df}{dt}(p, 0)\nabla_X F(\Omega, p)\,dp + \frac{d}{dp}\int_{\Omega_t}\tilde{\varphi}(p)\tilde{f}(p)\nabla_X F(\Omega_t, p)\,dp|_{t=0}$$

$$= \nabla_{\varphi\cdot X(f)\cdot X}F(\Omega) + \nabla_X\nabla_{\tilde{\varphi}\tilde{f}X}F(\Omega).$$

Proof of lemma 9. Taking covering $\{O_\lambda\}$ of $M$ by domains of coordinate charts $(\varkappa_\lambda, O_\lambda)$ and a subordinated partition of unity $\{\varphi_\lambda\}$, we can reduce our problem to the problem of smoothness of measures $\mu_\lambda := \mu\cdot\varphi_\lambda$. Let us define $\nu_\lambda := \varkappa_\lambda(\mu\cdot\varphi_\lambda)$. Now $\nu_\lambda$ is a finite measure in $R^m$ and the only thing which remains to prove is that all $\nu_\lambda$ are smooth $m$--forms.

Let us take the Fourier transform $\hat{\nu}_\lambda$. Because $(\pounds_X)^s\,\nu_\lambda$ is a finite measure for every constant field $X$ in $R^m$, $\hat{\nu}_\lambda(p)\cdot(\sum a_i p^i)^s$ is borned for every linear form $\sum a_i p^i$ on $R^m$. It means that $\hat{\nu}_\lambda(p)\cdot p^a$ is borned for every polynomial $p^a$ in $R^m$. The same arguments can be used for the finite measure $\nu_\lambda(x)\cdot x^\beta$ (for any polynomial $x^\beta$), and we infer that $p^a D^\beta \hat{\nu}_\lambda(p)$ is borned. It means that $\hat{\nu}_\lambda \in \mathscr{S}(R^m)$ (the Schwartz space) and also $\nu_\lambda \in \mathscr{S}(R^m)$. The last relation implies that $\nu_\lambda$ is smooth.

## APPENDIX

### ON A POSSIBILITY OF INTRODUCING A DIFFERENTIABLE STRUCTURE IN $\mathscr{P}_k$

The topology of compact convergence plays a fundamental role in the present paper. With its help we have defined smoothness of functionals with respect to 0-norms. As it was shown in [3], this topology in $\mathscr{P}_k$ cannot be equipped with a differentiable structure. Also topologies of uniform convergence of all derivatives up to $N$-th order have no differentiable structure. It is so, because differential operators are not continuous in the space $C^N$ for $N < \infty$. But they become continuous on $C^\infty$ and this fact has been used in [3] for defining a differentiable structure on $\mathscr{P}_k$ equipped with the topology of uniform convergence of all derivatives.

These reasons for resignation of a differentiable structure in $\mathscr{P}_k$ may be called local. However, global causes are equally serious. They are connected with the fact that we want to consider also non-compact submanifolds. With the lack of a uniform structure in $M$ the only natural topology is the topology of compact convergence. But in that case the

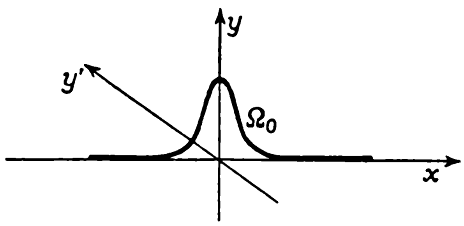


Fig. 2

method of parametrizing submanifolds lying in a neighbourhood of $\Omega$ by sections of a vector bundle deludes. We will demonstrate this on the following example:

Take $M = R^2$, $\Omega = \{(x, 0) \in R^2 : x \in R\}$. Let $\Omega_0$, as drawn on Fig. 2, be a smooth 1-dimensional submanifold.

If we take $\Omega_s := \{(x, y) : (x-s, y) \in \Omega_0\}$, then, obviously, every $\Omega_s$ can be treated as a section of the vector bundle whose fibres are parallel to the $y$-axis. Of course, $\Omega_s \to \Omega$ almost uniformly with all derivatives. But if we treat $M$ as a bundle with fibres parallel to $y'$-axis, then none of these submanifolds is a section. So, if we want to number submanifolds from $\mathscr{P}_k$ by sections of vector bundles, such "coordinate charts" would not cover entire neighbourhoods of points in $\mathscr{P}_k$.

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