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THREE COUNTABLE CONNECTED SPACES

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In [7] Cvid poses the question whether there exists a countable connected Hausdorff space such that the intersection of every two of its connected subsets is connected. A connected space having this property is called *strongly unicoherent.*

We shall construct three countable connected spaces with this property. The first is Hausdorff, the second Urysohn, and the third Urysohn almost regular. None of them is locally connected. Their construction is based on a modification of the embedding applied in [11] and on three auxiliary spaces which are countable Hausdorff, Urysohn, almost regular, respectively, and have two points a, b not separated by a continuous real-valued function. The third space has the additional property that it is regular at the points a, b.

For spaces countable connected or locally connected Hausdorff, Urysohn or almost regular or ones with other properties, see [1]–[31].

A topological space T is called

(1) *Urysohn* if every two points of T have disjoint closed neighbourhoods,

(2) *almost regular* if it has a dense subset at each point of which T is regular.

Let T be a connected topological space. A point t is called a *cut point* if the space $T\setminus\{t\}$ is not connected. Thus, if t is a cut point of T, then the subspace $T\setminus\{t\}$ is the union of two mutually separated sets $A(t)$, $B(t)$. (Two sets A, B are called *separated* if $A\cap\bar{B}=\emptyset$ and $\bar{A}\cap B=\emptyset$.) Obviously, if $A(t)$, $B(t)$ are connected, the separation is unique. Let $x, y\in T$. A cut point t of T is said to *separate* the points x, y if the above sets $A(t)$, $B(t)$ can be chosen so that $x\in A(t)$ and $y\in B(t)$. The set of cut points separating the points x, y will be denoted by $E(x, y)$. It is clear that if M is a connected subset of a (connected) space T and $x, y\in M$, then $E(x, y)\subseteq M$. (For these definitions see [32], p. 41.)

1. In the space of real numbers with the usual topology we consider the subspace

$$X = Z^+ \cup \{k \pm 1/m\colon\ k \in Z^+,\ m = 3, 4, \ldots\},$$

where Z^+ is the set of positive integers. We add two more points a, b and we set $S = X \cup \{a, b\}$. For the points a, b a basis of open neighbourhoods are the sets

$$B(a) = \{V_n(a) = \{a\} \cup \{k - 1/m\colon\ m = 3, 4, \ldots;\ k > n-1\}\colon\ n = 1, 2, \ldots\},$$

$$B(b) = \{V_n(b) = \{b\} \cup \{k + 1/m\colon\ m = 3, 4, \ldots;\ k > n-1\}\colon\ n = 1, 2, \ldots\}.$$

The space S is Hausdorff with the two points a, b not separated by disjoint closed neighbourhoods (because $\bar{V}_n(a) \cap \bar{V}_n(b) = \{n, n+1, \ldots\}$), and hence, for every continuous real-valued function f of S, $f(a) = f(b)$. The space S is due to Urysohn [30].

Let Z^- be the set of negative integers. In $Z^- \cup S$ consider each point of Z^- to be isolated. Obviously, the set

$$D = Z^- \cup \{k \pm 1/m\colon\ k \in Z^+,\ m = 3, 4, \ldots\}$$

consisting of isolated points of $Z^- \cup S$ is dense.

Set $J = Z^- \cup S \setminus \{a, b\}$. The natural linear order $\leqslant$ of the set of real numbers induces a linear order on J. For every two points x, y of J we set

$$L(x \leqslant) = \{z \in J\colon\ x \leqslant z\}, \qquad L(\leqslant x) = \{z \in J\colon\ z \leqslant x\},$$

$$L(x, y] = \{z \in J\colon\ x < z \leqslant y\}.$$

Similarly we define the sets $L(x <)$, $L(< x)$, $L[x, y)$, $L(x, y)$ and $L[x, y]$.

Definition. Two points x, $y \in J$ are said to be *successive* if $L(x, y) = \emptyset$.

Following the order on J we attach disjoint copies of J (that is, spaces homeomorphic to J) to every pair of isolated and successive points of J and we construct the set

$$T_1(J) = J \cup \bigcup_{\lambda \in \Lambda} J^\lambda,$$

where $\Lambda = \{(d_1, d_2) \in D \times D\colon\ d_1 < d_2,\ d_1, d_2 \text{ successive}\}$. It should be noted that to each isolated point of J there are attached two copies of J, to every successive pair of points there is attached only one copy, and to each $z \in Z^+$ no copy is attached.

For every two points x, y of J we set

$$\Lambda(x \leqslant) = \{\lambda = (d_1, d_2) \in \Lambda\colon\ x \leqslant d_1\},$$

$$\Lambda(\leqslant x) = \{\lambda = (d_1, d_2) \in \Lambda\colon\ d_2 \leqslant x\},$$

$$\Lambda(x, y] = \{\lambda = (d_1, d_2) \in \Lambda\colon\ x < d_1 < d_2 \leqslant y\}.$$

Similarly we define the sets $\Lambda(x <)$, $\Lambda(< x)$, $\Lambda[x, y]$, $\Lambda[x, y)$, and $\Lambda(x, y)$.

It is obvious that if $x \in Z^+$, then $\Lambda(x \leqslant) = \Lambda(x <)$ and $\Lambda(\leqslant x) = \Lambda(< x)$. If $x \in Z^+$ and $y \notin Z^+$, then $\Lambda(x, y] = \Lambda[x, y]$, and if $x, y \in Z^+$, then $\Lambda[x, y] = \Lambda(x, y)$.

Finally, for every point $x \in J$ we set

$$T_1(L(x \leqslant)) = L(x \leqslant) \cup \bigcup_{\lambda \in \Lambda(x \leqslant)} J^\lambda,$$

$$T_1(L(\leqslant x)) = L(\leqslant x) \cup \bigcup_{\lambda \in \Lambda(\leqslant x)} J^\lambda,$$

$$T_1(L(x, y]) = L(x, y] \cup \bigcup_{\lambda \in \Lambda(x, y]} J^\lambda.$$

Similarly we define the sets $T_1(L(x <))$, $T_1(L(< x))$, $T_1(L[x, y])$, $T_1(L[x, y))$ and $T_1(L(x, y))$.

On the set $T_1(J)$ we define the following topology:

On each copy J^λ the topology remains the same as in J. For every isolated point t of J a basis of open neighbourhoods of t in $T_1(J)$ is the set

$$B(t) = \{O_n(t) = \{t_n\} \cup (H_n(t) \cup G_n(t)) \colon n = 1, 2, \ldots\},$$

where $H_n(t)$, $G_n(t)$ are copies of the deleted neighbourhoods of the points a, b, respectively, attached to t. And for every $z \in Z^+$ a basis of open neighbourhoods is the set

$$B(z) = \{O_n(z) = V_n(z) \cup (\bigcup_{t \in V_n(z) \setminus \{z\}} (H_n(t) \cup G_n(t))) \colon n = 1, 2, \ldots\},$$

where $V_n(z)$, $n = 1, 2, \ldots$, are open neighbourhoods of the point z in the space S.

It can be easily verified that the space $T_1(J)$ is Hausdorff.

We prove that $T_1(J)$ has the following properties:

(P.1) *Every continuous real-valued function of $T_1(J)$, restricted to J, is constant.*

(P.2) *Every continuous real-valued function of each of the spaces $T_1(L(x \leqslant))$, $T_1(L(\leqslant x))$, $T_1(L(x <))$, $T_1(L(< x))$, $T_1(L(x, y])$, $T_1(L[x, y])$, and $T_1(L(x, y))$, restricted to $L(x \leqslant)$, $L(\leqslant x)$, $L(x <)$, $L(< x)$, $L(x, y]$, $L[x, y]$, and $L(x, y)$, respectively, is constant.*

(P.3) *If x, y are non-successive points of J, then the sets $L(< z)$, $L(z <)$, $z \in L(x, y)$, are separated by a continuous real-valued function of $T_1(J) \setminus \{z\}$.*

Proof. (P.1) Every two points x, y of J are either successive or not. If they are successive (hence isolated in J), there exists a copy $J^\lambda \subsetneqq T_1(J)$, $\lambda = (x, y)$, such that

$$\bar{J}^\lambda = J^\lambda \cup \{x, y\},$$

which implies that, for every continuous real-valued function f of $T_1(J)$, $f(x) = f(y)$. If the points $x, y, x < y$, are not successive, then we consider the set

$$T_1(L[x, y]) = L[x, y] \cup \bigcup_{\lambda \in \Lambda[x,y]} J^{\lambda}.$$

It can be easily proved that every continuous real-valued function of $T_1(J)$ restricted to $D \cap L[x, y]$ is constant, and since $D \cap L[x, y]$ is dense in $L[x, y]$, it follows that f is constant on $L[x, y]$, i.e., $f(x) = f(y)$.

(P.2) It is proved in the same way as (P.1).

(P.3) Since $z \in L(x, y)$, either z is isolated in J or $z \in Z^+ \cap L(x, y)$. If z is isolated, then for the copies J^{λ_1}, J^{λ_2} attached to z we have

$$\lambda_1 = (d_1, z), \quad \lambda_2 = (z, d_2) \quad \text{and} \quad \{z\} = \bar{J}^{\lambda_1} \cap \bar{J}^{\lambda_2}.$$

If $z \in Z^+ \cap L(x, y)$, then no copy is attached to z. In both cases the sets

$$L(z <) \cup \bigcup_{\lambda \in \Lambda(z <)} J^{\lambda} \quad \text{and} \quad L(< z) \cup \bigcup_{\lambda \in \Lambda(< z)} J^{\lambda}$$

are open-and-closed in $T_1(J) \setminus \{z\}$, and hence the characteristic function separates them in $T_1(J) \setminus \{z\}$. Therefore, it separates their subsets $L(z <)$ and $L(< z)$, respectively.

After we constructed the space $T_1(J)$ and following the linear order on each copy J^{λ}, $\lambda \in \Lambda$, of the space $T_1(J)$, we attach disjoint copies of J on each copy J^{λ}, $\lambda \in \Lambda$, in the same manner as we did for the construction of $T_1(J)$. Thus, we construct the space $T_2(J) = T_1(T_1(J))$, and then, by induction, the space $T_n(J) = T_1(T_{n-1}(J))$.

We set

$$T(J) = \bigcup_{n=0}^{\infty} T_n(J),$$

where $T_0(J) = J$, and on $T(J)$ we define a topology as in [11], Section 4, for $I(X)$.

In the sequel, to simplify the notation we will write T and T_n, $n = 0, 1, \ldots$, in place of $T(J)$ and $T_n(J)$, $n = 0, 1, \ldots$, respectively, and if no confusion is alike, the subsets of the space J will be identified with their copies in J^{λ}.

PROPOSITION. *The space T has the following properties:*

(T.1) *It is Hausdorff connected.*

(T.2) *For every copy J^{λ} and every $t \in J^{\lambda}$, if n is the minimal integer for which $J^{\lambda} \subsetneq T_n$, the sets $T(L(< t))$ and $T(T_n \setminus L(< t))$ are connected.*

(T.3) *Every point of T is a cut point, cutting the space T into connected subspaces.*

(T.4) *If $x, y \in T$ and x, y are not successive, then $E(x, y) \neq \emptyset$.*

(T.5) *For every two connected subsets M, N of T the set $M \cap N$ is connected.*

Proof. (T.1) That T is Hausdorff follows directly from the definition of the topology on T. In order to prove that T is connected it suffices to show that if x, y are two arbitrary points of T, then $f(x) = f(y)$ for every continuous real-valued function f of T. Let x, y be two points of T. If they are not successive, we have the following cases:

(1) Both x, y belong to a common copy J^λ and $L(x, y) \neq \emptyset$.

(2) The point y belongs to a copy $J^\lambda \subsetneqq T_n$ for some integer n, but $x \in \bar{J}^\lambda \setminus J^\lambda$.

(3) The points x, y belong to arbitrary copies $J^{\mu 0}$, $J^{\nu 0}$, respectively, and $x \notin \bar{J}^{\nu 0}$, $y \notin \bar{J}^{\mu 0}$.

The cases where x, y are successive (and hence they belong to the same copy J^λ) or they belong to the same copy J^λ, but $L(x, y) \neq \emptyset$, are proved in the same way as (P.1). For if n is the minimal integer such that $J^\lambda \subsetneqq T_n$, then the space $T_{n+1}(J^\lambda)$ is homeomorphic to the space $T_1(J)$.

For case (2) we first consider the set

$$L(y \leqslant) = \{z \in J^\lambda \colon y \leqslant z\} \subsetneqq T_n,$$

and then the set

$$L(y \leqslant) \cup \Big(\bigcup_{\lambda \in \Lambda(y \leqslant)} J^\lambda \Big) \cup \{x\}$$

which is the required set joining the points x, y.

For case (3) let m, n, $m \leqslant n$, be the minimal integers for which $x \in T_m$, $y \in T_n$ and let $J^{\mu 0}$, $J^{\nu 0}$ be the copies containing the points x, y, respectively. Consider the set

$$L_0(y) = L(y \leqslant) \cup \bigcup_{\lambda \in \Lambda(y \leqslant)} J^\lambda .$$

The copy $J^{\nu 0}$ is attached to a unique pair of successive points y_1', y_1 of a copy $J^{\nu 1}$ in the subspace T_{n-1}. Obviously,

$$\bar{J}^{\nu 1} = J^{\nu 1} \cup \{y_1', y_1\}.$$

Assume that $y_1' < y_1$ and consider the set

$$L_1(y) = L(y_1 \leqslant) \cup \bigcup_{\lambda \in \Lambda(y_1 \leqslant)} J^\lambda .$$

Similarly, the copy $J^{\nu 1}$ is attached to a unique pair of points y_2', y_2 of a copy $J^{\nu 2}$ in the subspace T_{n-2} for which

$$\bar{J}^{\nu 2} = J^{\nu 2} \cup \{y_2', y_2\}.$$

Assume $y_2' < y_2$ and consider the set

$$L_2(y) = L(y_2 \leqslant) \cup \bigcup_{\lambda \in \Lambda(y_2 \leqslant)} J^\lambda.$$

Continuing in this manner we find a copy $J^{\nu_{n-m+1}}$ and two points y_{n-m+1}', y_{n-m+1} such that $y_{n-m+1}', y_{n-m+1} \in J^{\nu_{n-m+1}} \subsetneqq T_{m+1}$. We set

$$L_{n-m+1}(y) = L(y_{n-m+1} \leqslant) \cup \bigcup_{\lambda \in \Lambda(y_{n-m+1} \leqslant)} J^\lambda$$

and let y_{n-m}', y_{n-m} be the points of T_m to which the copy $J^{\nu_{n-m+1}}$ is attached (that is, $\bar{J}^{\nu_{n-m+1}} = J^{\nu_{n-m+1}} \cup \{y_{n-m}', y_{n-m}\}$).

For the points x, y_{n-m}', y_{n-m} of T_m we have three cases:

(3a) They are distinct and belong to the same copy.

(3b) The point x coincides with one of the points y_{n-m}', y_{n-m}.

(3c) The point x belongs to a different copy than the points y_{n-m}', y_{n-m}.

It is obvious that for case (3a) the required set joining the points x, y is

$$\bigcup_{i=0}^{n-m+1} L_i(y) \cup L[x, y_{n-m}'] \cup \bigcup_{\lambda \in \Lambda[x, y_{n-m}']} J^\lambda \quad \text{if } x < y_{n-m}'$$

or

$$\bigcup_{i=0}^{n-m+1} L_i(y) \cup L[y_{n-m}, x] \cup \bigcup_{\lambda \in \Lambda[y_{n-m}, x]} J^\lambda \quad \text{if } y_{n-m} < x,$$

and for case (3b) the set

$$\bigcup_{i=0}^{n-m+1} L_i(y) \cup \{x\} \quad \text{if } x = y_{n-m}' \text{ or } x = y_{n-m}.$$

For case (3c), where $x \in J^{\mu_0}$, $y_{n-m}', y_{n-m} \in J^{\nu_{n-m}}$ and $\mu_0 \neq \nu_{n-m}$, we repeat the above process for the point x and in a "parallel" way for the point y_{n-m} (if $y_{n-m}' < y_{n-m}$) or for the point y_{n-m}' (if $y_{n-m} < y_{n-m}'$). Thus we find a common copy $J^t \subsetneqq T_k$, $k < m$, where the pairs x_{m-k}', x_{m-k} and y_{n-k}', y_{n-k} corresponding to x, y, respectively, belong. Since the points x_{m-k}', x_{m-k} and y_{n-k}', y_{n-k} are successive, either are all distinct or two of them coincide. Hence, setting

$$A = \bigcup_{i=0}^{n-k+1} L_i(y), \quad B = \bigcup_{i=0}^{m-k+1} L_i(x),$$

it is obvious that the required sets are the following:

$$A \cup B \cup L[x_{m-k}, y_{n-k}'] \cup \bigcup_{\lambda \in \Lambda[x_{m-k}, y_{n-k}']} J^\lambda \quad \text{if } x_{m-k} < y_{n-k}',$$

$$A \cup B \cup L[y_{n-k}, x_{m-k}'] \cup \bigcup_{\lambda \in \Lambda[y_{n-k}, x_{m-k}']} J^\lambda \quad \text{if } y_{n-k} < x_{m-k}',$$

$$A \cup B \cup \{x_{m-k}\} \quad \text{if } x_{m-k} = y_{n-k}',$$

$$A \cup B \cup \{y_{n-k}\} \quad \text{if } y_{n-k} = x_{m-k}'.$$

(T.2) It is proved in exactly the same way as (T.1).

(T.3) Let $t \in T$ and n be the minimal integer for which $t \in T_n$. There exists a copy $J^\lambda \subsetneq T_n$ such that $t \in J^\lambda$. Consider the subspaces $L(<t)$ and $T_n \setminus L(<t)$ of T_n. By the construction of T_{n+1}, for $T_{n+1}(L(<t))$ and $T_{n+1}(T_n \setminus L(<t))$ we have

$$\overline{T_{n+1}(L(<t))} \cap \overline{T_{n+1}(T_n \setminus L(<t))} \neq \{t\}.$$

Consider now the connected subspaces $T(L(<t))$ and $T(T_n \setminus L(<t))$ of T. By the construction of T it follows that

$$\overline{T(L(<t))} \cap \overline{T(T_n \setminus L(<t))} = \{t\}.$$

(T.4) Since the points x, y are not successive, we consider the three cases described in (T.1) above. For case (1) it is obvious that $E(x, y) = L(x, y)$, and for case (2) that $E(x, y) = L(y <)$.

For case (3), which is divided into three subcases, the required sets are: for (3a),

$$E(x, y) = \begin{cases} L(y <) \cup \bigcup_{i=1}^{n-m+1} L(y_i \leqslant) \cup L(x, y'_{n-m}) & \text{if } x < y'_{n-m}, \\ L(y <) \cup \bigcup_{i=1}^{n-m+1} L(y_i \leqslant) \cup L[y_{n-m}, x) & \text{if } y_{n-m} < x; \end{cases}$$

for (3b),

$$E(x, y) = L(y <) \cup \bigcup_{i=1}^{n-m+1} L(y_i \leqslant);$$

for (3c),

$$E(x, y) = \begin{cases} A' \cup B' \cup L[x_{m-k}, y'_{n-k}] & \text{if } x_{m-k} < y'_{n-k}, \\ A' \cup B' \cup L[y_{n-k}, x_{n-k}] & \text{if } y_{n-k} < x'_{m-k}, \\ A' \cup B' \cup \{x_{m-k}\} & \text{if } x_{m-k} = y'_{n-k}, \\ A' \cup B' \cup \{y_{n-k}\} & \text{if } y_{n-k} = x'_{m-k}; \end{cases}$$

where

$$A' = L(y <) \cup \bigcup_{i=1}^{n-m+1} L(y_i \leqslant) \quad \text{and} \quad B' = L(x <) \cup \bigcup_{i=1}^{n-m+1} L(x_i \leqslant).$$

(T.5) Let M, N be two connected subsets of T and $x, y \in M \cap N$. If x, y are successive, there exist a copy J^λ and an integer n such that $x, y \in \bar{J}^\lambda$, $\lambda = (x, y)$ and $J^\lambda \subseteq T_n$. Since every continuous real-valued function of the connected subset M is constant, there exists a point $k \in (Z^+)^\lambda \subseteq J^\lambda$ such that $\{k, k+1, \ldots\} \subseteq M$. Similarly, for the connected subset N there exists $l \in (Z^+)^\lambda \subseteq J^\lambda$ such that $\{l, l+1, \ldots\} \subseteq N$. Hence for the set of cut points separating the points k, x (or k, y) of M and the points l, x (or l, y) of N we have

$$E(k, x) = E(k, y) = L(k <) \subseteq M \quad \text{and} \quad E(l, x) = E(l, y) = L(l <) \subseteq N.$$

Setting $m = \max\{k, l\}$ we obtain $L(m <) \subseteq M \cap N$, which means that the successive points x, y cannot be separated by disjoint open-and-closed subsets in $L(m <) \cup \{x, y\}$, and therefore cannot be separated by disjoint open-and-closed subsets in $M \cap N$.

If x, y are not successive, then every pair of successive points of $E(x, y) \cup \{x, y\} \subseteq M \cap N$, cannot be separated by disjoint open-and-closed subsets in $M \cap N$, and hence every pair of points of $E(x, y) \cup \{x, y\}$ cannot be separated. Therefore, in both cases the set $M \cap N$ is connected.

2. We now construct first a countable connected Urysohn space such that the intersection of every pair of connected subsets is connected, and then a countable connected Urysohn almost regular space with this property. For the first space we need to construct a countable Urysohn space Y having two points a, b such that $f(a) = f(b)$ for every continuous real-valued function f of Y, and for the second, a countable Urysohn almost regular space $\boldsymbol{R}$ having two points a, b such that $\boldsymbol{R}$ is regular at a, b and $f(a) = f(b)$ for every continuous real-valued function f of $\boldsymbol{R}$. Then, using the method of Section 1 we shall construct in each case first the space T_1, and then the spaces $T_2, \ldots, T_n, \ldots$ and T. This final space T will be the required one.

(A) In the space of real numbers with the usual topology we consider the subspaces

$$X_1 = Z^+ \cup \{k \pm 1/m\colon\ k \in Z^+,\ m = 3, 4, \ldots\},$$

$$X_2 = Z^- \cup \{k \pm 1/m\colon\ k \in Z^-,\ m = 3, 4, \ldots\},$$

where Z^+ and Z^- are the positive and negative integers, respectively. In the topological sum $X_1 \cup X_2$ we add one more point p and we set

$$X = X_1 \cup \{p\} \cup X_2.$$

A basis of open neighbourhoods for the point p is

$$B(p) = \{V_n(p) = \{p\} \cup A_n^+ \cup A_n^-\colon\ n = 1, 2, \ldots\},$$

where

$$A_n^+ = \bigcup_{k \geqslant 2n} \{k - 1/m\colon\ m = 3, 4, \ldots\} \cup \bigcup_{k \geqslant 2n-1} \{k + 1/m\colon\ m = 3, 4, \ldots\},$$

$$A_n^- = \bigcup_{k \leqslant -2n+1} \{k - 1/m\colon\ m = 3, 4, \ldots\} \cup \bigcup_{k \leqslant -2n} \{k + 1/m\colon\ m = 3, 4, \ldots\}.$$

Observe that the set $D = \{k \pm 1/m\colon\ k \in Z^+ \cup Z^-,\ m = 3, 4, \ldots\}$ consisting of the isolated points of X is dense.

Let $X^{(1)}, X^{(2)}, \ldots, X^{(n)}, \ldots$ be disjoint copies of the space X. In the topological sum $\bigcup_{n=1}^{\infty} X^{(n)}$ we add two more points a, b and we set

$$Y = \Big(\bigcup_{n=1}^{\infty} X^{(n)}\Big) \cup \{a, b\}.$$

In order to define a basis of open neighbourhoods for the point a we first consider the sets

$$A^+ = \{2n+1/m\colon\ n \in N_1,\ m = 3, 4, \ldots\} \cup \{2n-1-1/m\colon\ n \in N_2,\ m = 3, 4, \ldots\},$$

$$A^- = \{-2n-1/m\colon\ n \in N_1,\ m = 3, 4, \ldots\} \cup \{-2n+1+1/m\colon\ n \in N_2,\ m = 3, 4, \ldots\},$$

where N_1 and N_2 are the odd and even positive integers, respectively. Set $A = A^+ \cup A^-$ and let $A^{(k)}$ be the copy of A in $X^{(k)}$. Then a basis of open neighbourhoods for the point a is

$$B(a) = \Big\{V_n(a) = \{a\} \cup \Big(\bigcup_{k=n}^{\infty} A^{(k)}\Big)\colon\ n = 1, 2, \ldots\Big\}.$$

Similarly, for the point b we have

$$B^+ = \{2n-1-1/m\colon\ n \in N_1,\ m = 3, 4, \ldots\} \cup \{2n+1/m\colon\ n \in N_2,\ m = 3, 4, \ldots\},$$

$$B^- = \{-2n+1+1/m\colon\ n \in N_1,\ m = 3, 4, \ldots\} \cup \{-2n-1/m\colon\ n \in N_2,\ m = 3, 4, \ldots\}$$

and $B = B^+ \cup B^-$. If $B^{(k)}$ is the copy of B in $X^{(k)}$, then a basis of open neighbourhoods for the point b is

$$B(b) = \Big\{V_n(b) = \{b\} \cup \Big(\bigcup_{k=n}^{\infty} B^{(k)}\Big)\colon\ n = 1, 2, \ldots\Big\}.$$

It can be easily proved that Y is Urysohn and $f(a) = f(b)$ for every continuous real-valued function f of Y. Observe that the subset $\bigcup_{n=1}^{\infty} D^{(n)}$ ($D^{(n)}$ is the copy of D in $X^{(n)}$) of the isolated points of Y is dense.

COROLLARY 1. *There exists a countable connected Urysohn space in which the intersection of every pair of connected subsets is connected.*

Proof. Let Y be the space constructed above and Z^- be the set of negative integers. In $Z^- \cup Y$ we consider each point of Z^- to be isolated.

Set $J = Z^- \cup Y \setminus \{a, b\}$ and define a linear order $\leqslant^*$ on J as follows:

If $x, y \in Z^-$ and $x \leqslant y$ (in the natural linear order $\leqslant$ of the set of real numbers), then $x \leqslant^* y$. If $x^{(n)} \in X^{(n)}$, $x^{(m)} \in X^{(m)}$ and $n \leqslant m$, then

$$x^{(n)} \leqslant^* x^{(m)}.$$

If $x^{(n)}, y^{(n)} \in X^{(n)}$ $(n = 1, 2, \ldots)$, $x^{(n)}, y^{(n)} \in X_1^{(n)}$ (or $x^{(n)}, y^{(n)} \in X_2^{(n)}$) and $x \leqslant y$ in

X_1 (or in X_2), then

$$x^{(n)} \leqslant^* y^{(n)}.$$

And if $z \in Z^-$, $x^{(n)} \in X_1^{(n)}$, $y^{(n)} \in X_2^{(n)}$ and $p^{(n)}$ is the copy of p in $X^{(n)}$, then

$$z \leqslant^* x^{(n)} \leqslant^* p^{(n)} \leqslant^* y^{(n)}.$$

Following this linear order we construct the spaces

$$T_1(J),\ T_2(J), \ldots, T_n(J), \ldots, \qquad T(J) = \bigcup_{n=0}^{\infty} T_n(J)$$

in the same manner as in Section 1 (i.e., the copies of J are attached to the successive and isolated points, no copy is attached to (the copies of) Z^+, Z^-, p).

It is easy to check that T is Urysohn. The other properties of T are proved as for the space T of the Proposition in Section 1.

(B) In order to construct the second space we first consider the sets

$$M(n, k^-) = \{(n, k)\colon k = 0, -1, -2, \ldots, -n\},$$

$$M(n, k^+) = \{(n, k)\colon k = 0, 1, 2, \ldots, n\},$$

$$M(n, k) = \{(n, k)\colon k = 0, \pm 1, \pm 2, \ldots, \pm n\},$$

where $n = 1, 2, \ldots$ We set

$$M = \bigcup_{n=1}^{\infty} M(n, k)$$

and on M we define the following linear order: if $n \leqslant m$, then $(n, k) \leqslant (m, k)$. If n is odd and $k_1 \leqslant k_2$, then $(n, k_2) \leqslant (n, k_1)$. If n is even and $k_1 \leqslant k_2$, then $(n, k_1) \leqslant (n, k_2)$.

Following this linear order we attach disjoint copies of the space J of Corollary 1 to each pair of successive points of each set $M(n, k)$, $n = 1, 2, \ldots$, and we consider the topological sum

$$Q = \bigcup_{n=1}^{\infty} \Big(M(n, k) \cup \bigcup_{\lambda \in \Lambda(n,k)} J^{\lambda}\Big),$$

where

$$\Lambda(n, k) = \{(d_1, d_2) \in M(n, k) \times M(n, k)\colon d_1 < d_2,\ d_1, d_2 \text{ successive}\}.$$

It should be noted that for every $n = 1, 2, \ldots$ no copy is attached to the pairs

$$(2n-1, -(2n-1)), (2n, -2n) \quad \text{and} \quad (2n, 2n), (2n+1, 2n+1).$$

On the space Q we add two more points a, b and we set $R = Q \cup \{a, b\}$. In order to define a basis of open neighbourhoods of the points a, b we first consider the sets

$$Q^- = \bigcup_{n=1}^{\infty} \big(M(n, k^-) \cup \bigcup_{\lambda \in \Lambda(n,k^-)} J^\lambda\big), \quad Q^+ = \bigcup_{n=1}^{\infty} \big(M(n, k^+) \cup \bigcup_{\lambda \in \Lambda(n,k^+)} J^\lambda\big),$$

where

$$\Lambda(n, k^-) = \{(d_1, d_2) \in M(n, k^-) \times M(n, k^-)\colon d_1 < d_2, d_1, d_2 \text{ successive}\},$$

$$\Lambda(n, k^+) = \{(d_1, d_2) \in M(n, k^+) \times M(n, k^+)\colon d_1 < d_2, d_1, d_2 \text{ successive}\}.$$

Then for $m = 1, 2, \ldots$ we set

$$A_m = \{J^{\lambda_n^m} \cup \{(n, -m)\}\colon \lambda_n^m = ((n, m), (n, -m+1)), n = m, m+1, \ldots\},$$

$$B_m = \{J^{\lambda_n^m} \cup \{(n, m)\}\colon \lambda_n^m = ((n, m), (n, m-1)), n = m, m+1, \ldots\}.$$

A basis of open neighbourhoods of the points a, b are the sets

$$B(a) = \{V_m(a) = \{a\} \cup (X^- \setminus \bigcup_{i=0}^{m-1} A_i)\colon m = 1, 2, \ldots\},$$

$$B(b) = \{V_m(b) = \{b\} \cup (X^+ \setminus \bigcup_{i=0}^{m-1} B_i)\colon m = 1, 2, \ldots\},$$

where $A_0 = B_0 = \{(n, 0)\colon n = 1, 2, \ldots\}$.

It can be easily proved that $\boldsymbol{R}$ is Urysohn almost regular and $f(a) = f(b)$ for every continuous real-valued function f of $\boldsymbol{R}$. To prove that $\boldsymbol{R}$ is regular at a, b observe that, for $m = 1, 2, \ldots$,

$$\bar{V}_{m+1}(a) = V_{m+1}(a) \cup \{(n, -m)\colon n = m+1, m+2, \ldots\} \subseteq V_m(a),$$

$$\bar{V}_{m+1}(b) = V_{m+1}(b) \cup \{(n, m)\colon n = m+1, m+2, \ldots\} \subseteq V_m(b).$$

COROLLARY 2. *There exists a countable connected Urysohn almost regular space in which the intersection of every pair of connected subsets is connected.*

Proof. Let $\boldsymbol{R}$ be the space constructed above and Z^- be the set of negative integers. In $Z^- \cup \boldsymbol{R}$ consider each point to be isolated.

On $Z^- \cup M$ we define the following linear order $\leqslant$: On Z^- the order is the induced natural linear order of the set of real numbers. On the set M the order is as defined in (B) above. If $z \in Z^-$ and $m \in M$, we set $z \leqslant m$.

We set $J' = Z^- \cup \boldsymbol{R} \setminus \{a, b\}$ and following the order on Z^- and on the pairs of points of M where no copy of J was attached, i.e.,

$$\begin{aligned} \Lambda = &\{(z_1, z_2) \in Z^- \times Z^-\colon z_1 < z_2, z_1, z_2 \text{ successive}\} \\ &\cup \{((2n-1, -(2n-1)), (2n, -2n))\colon n = 1, 2, \ldots\} \\ &\cup \{((2n, 2n), (2n+1, 2n+1))\colon n = 1, 2, \ldots\}, \end{aligned}$$

we construct the space

$$T_1(J') = J' \cup \bigcup_{\lambda \in \Lambda} J'^{\lambda}$$

as in Section 1.

Repeating this attaching to each (attached) copy of J' and attaching copies of J' to each copy of J (following the linear order $\leqslant^*$ on J defined in Corollary 1) we construct the spaces $T_2(J'), \ldots, T_n(J'), \ldots$ and

$$T(J') = \bigcup_{n=0}^{\infty} T_n(J').$$

That $T(J')$ is almost regular follows from the fact that $\boldsymbol{R}$ is regular at a, b and the copies of J' are attached to the dense subset of isolated points. The other properties of $T(J')$ are proved as in the Proposition of Section 1.

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