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ON RANK OF EXTENSIONS OF VALUATIONS

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0. Introduction. In this paper, our main aim is to prove the following

THEOREM 1. *Let v_0 be a valuation of finite rank r of a field K_0. Let K be an overfield of finite transcendence degree s over K_0. If V is an extension of v_0 to K, then the rank of V does not exceed $r+s$.*

The above result is well known ([1], Section 10.3, Corollary 2, Theorem 1; [2], p. 239, Theorem 17.7; [3], p. 116, Theorem 5.24). However, our method of proof is different from the standard one and perhaps new.

We also construct explicitly an extension of any given valuation v_0 of rank r to the field $K_0(x_1, x_2, \ldots, x_s)$ of s independent variables having the rank $r+s$.

1. Proof of Theorem 1. Theorem 1 is equivalent to the following

THEOREM 1′. *Let v_0 be a valuation of finite rank r of a field K_0 and let $K = K_0(x)$ be a simple transcendental extension of K_0. If V is an extension of v_0 to K, then the rank of V does not exceed $r+1$.*

In the proof of the following lemma and of Theorem 1′, we shall regard equivalent valuations as equal.

LEMMA 1. *Let v_0 be a valuation of rank 1 of a field K_0 with residue field k_0 and let $K = K_0(x)$ be a simple transcendental extension of K_0. If V is an extension of v_0 to K with residue field Δ, then either rank of V is 1 or rank of V is 2 with Δ algebraic over k_0 in the second case.*

Proof. Suppose if possible that there exists an extension V of v_0 to K with $\operatorname{rank} V > 2$. Thus there exist valuations V_1 and V_2 of K whose valuation rings satisfy

$$(1.1) \qquad R_V \subsetneqq R_{V_1} \subsetneqq R_{V_2} \subsetneqq K.$$

Let $\bar{V}_1$ be a valuation of the residue field of V_1, and $\bar{V}_2$ and $\bar{W}_2$ be valuations of the residue field of V_2 such that

$$(1.2) \qquad V = V_1 \circ \bar{V}_1, \quad V = V_2 \circ \bar{V}_2, \quad V_1 = V_2 \circ \bar{W}_2.$$

Since v_0 has rank 1 and since $R_V \cap K_0 = R_{v_0}$ is a maximal subring of K_0, it follows from (1.1) that $R_{V_1} \cap K_0$ is either K_0 or R_{v_0}. We show that the first case cannot occur, for otherwise V_1 would be trivial on K_0, which implies that V_1 has rank 1 and the valuation $\bar{V}_1$ defined on the residue field of V_1 (which is a finite extension of K_0), being an extension of v_0, has the same rank as v_0, that is 1. Thus

$$\operatorname{rank} V = \operatorname{rank} V_1 + \operatorname{rank} \bar{V}_1 = 2,$$

which contradicts our supposition. Therefore

$$R_{V_1} \cap K_0 = R_{v_0}.$$

The same argument may be applied to prove that

$$R_{V_2} \cap K_0 = R_{v_0}.$$

We claim that each of $\bar{V}_1$, $\bar{V}_2$ and $\bar{W}_2$ is of rank 1. This will give the desired contradiction, and thereby prove that the rank of any extension of v_0 to K does not exceed 2. For, by (1.2), $\bar{V}_2 = \bar{W}_2 \circ \bar{V}_1$. This would be impossible once it is proved that each of $\bar{V}_1$, $\bar{V}_2$, $\bar{W}_2$ has rank 1. To prove the above claim, it is enough to prove that if W and W' are two extensions of the valuation v_0 to K and if

$$R_W \underset{\neq}{\subset} R_{W'} \underset{\neq}{\subset} K,$$

then the valuation $\bar{W}'$ defined on the residue field Δ' of W', satisfying the relation $W = W' \circ \bar{W}'$, has rank 1. Here W and W' are extensions of v_0, so $\bar{W}'$ is trivial on k_0. Since the transcendence degree of K/K_0 is 1, therefore by Corollary 1 of [4], p. 25, the transcendence degree of Δ' over k_0 is $\leqslant 1$. Thus $\bar{W}'$, being a (non-trivial) valuation of Δ' over k_0, must have rank 1, which proves the assertion made above. We also remark that with W and W' as above, the residue field of W, i.e., the residue field of $\bar{W}'$, is algebraic over k_0. For either the residue field Δ' of W' is algebraic over k_0 or Δ' is an algebraic extension of a simple transcendental extension, say $k_0(y)$, of k_0. In both cases, it is clear that the residue field of the valuation $\bar{W}'$, defined on Δ' and trivial on k_0, is an algebraic extension of k_0.

We now prove the second assertion of the lemma; suppose that rank $V = 2$. We want to show that the residue field Δ of V is algebraic over k_0. Let V_1 be a valuation of rank 1 of K such that

$$R_V \underset{\neq}{\subset} R_{V_1} \underset{\neq}{\subset} K \tag{1.3}$$

and let $\bar{V}_1$ be the valuation of the residue field of V_1 such that $V = V_1 \circ \bar{V}_1$. Since v_0 is of rank 1, $R_{V_1} \cap K_0$ is either R_{v_0} or K_0. In the first case, both V_1 and V are extensions of v_0 to K, so by the remark made at the end of the above paragraph, the residue field Δ of $\bar{V}_1$ is an algebraic extension of k_0. In the second case, i.e., where V_1 is trivial on K_0, the residue field Δ_1 of V_1 is a finite extension of K_0. Since $\bar{V}_1$ is an extension of v_0 to Δ_1, the residue field Δ of $\bar{V}_1$ is a finite extension of k_0 in this case. This completes the proof of the lemma.

Proof of Theorem 1′. As before the residue fields of v_0 and V will be denoted by k_0 and Δ, respectively. The theorem is proved by induction on r. If $r = 0$, the result is well known. The above lemma shows that the theorem holds also for $r = 1$. Assume that $r \geqslant 2$ and as the induction hypothesis suppose that the theorem holds for valuations of rank $\leqslant r-1$ defined on any field. Let v_0 be a valuation of K_0 of rank r. Write $v_0 = w \circ \bar{w}$, where w is a valuation of K_0 of rank 1 and $\bar{w}$ is a valuation of the residue field of w. Since V is an extension of v_0 to K, therefore by Lemma 4 in [4], p. 53, there exist extensions W^* and $\bar{W}^*$ of w and $\bar{w}$ to the field K and to the residue field of W^*, respectively, such that

$$V = W^* \circ \bar{W}^*.$$

Observe that by Corollary 1 of [4], p. 25, the transcendence degree of the residue field of W^* over that of w does not exceed 1. Thus, by the induction hypothesis applied to the valuations w and $\bar{w}$, we have

$$\operatorname{rank} W^* \leqslant 2, \qquad \operatorname{rank} \bar{W}^* \leqslant r. \tag{1.4}$$

If $\operatorname{rank} W^* = 2$, then by Lemma 1 the residue field of W^* is an algebraic extension of the residue field of w; consequently, in this case

$$\operatorname{rank} \bar{W}^* = \operatorname{rank} \bar{w} = r-1.$$

So at least one of the inequalities in (1.4) is strict. Hence

$$\operatorname{rank} V = \operatorname{rank} W^* + \operatorname{rank} \bar{W}^* \leqslant r+1.$$

The proof is now complete.

2. An extension of v_0 to $K_0(x_1, \ldots, x_s)$ of rank $r+s$. Let F be any field and let $F(x)$ be the field of rational functions in an indeterminate x. We shall denote by V_x the usual x-adic valuation of $F(x)$, i.e., the valuation defined by $V_x(f(x))$, the exact power of the monomial x dividing a non-zero polynomial $f(x)$ over F; and by f^* we denote the constant term of the polynomial $f(x)x^{-V_x(f)}$.

Lemma 2. *With the notation as above, if V^* is a valuation of F of (finite) rank t with value group G^*, then the mapping W defined by*

$$W(f(x)) = \big(V_x(f(x)),\, V^*(f^*)\big) \qquad (f(x) \neq 0 \text{ in } F[x])$$

gives a valuation on $F(x)$ with value group $Z \times G^*$ (lexicographically ordered) of rank $t+1$.

This lemma can be easily proved and we omit its proof.

Let v_0 be a valuation of rank r of a field K_0 with value group G_0. We construct an extension W_s of v_0 to the field $K = K_0(x_1, \ldots, x_s)$ of s independent variables. For $1 \leqslant i \leqslant s$, we denote by K_i the field $K_0(x_1, \ldots, x_i)$, and by V_{x_i} the x_i-adic valuation on $K_i = K_{i-1}(x_i)$ as defined at the beginning of the section. For any non-zero polynomial $f(x_1, \ldots, x_s)$ over K_0, we define the non-zero polynomials

$$f^{(1)}(x_1, \ldots, x_{s-1}), \ldots, f^{(s-1)}(x_1), f^{(s)}$$

inductively as follows:

Write $f^{(1)}(x_1, \ldots, x_{s-1})$ for the value of the polynomial

$$f(x_1, \ldots, x_s)x_s^{-V_{x_s}(f)} \quad \text{at } (x_1, \ldots, x_{s-1}, 0);$$

in other words, if one writes f as

$$f(x_1, \ldots, x_s) = \sum_i g_i(x_1, \ldots, x_{s-1})x_s^i$$

and if k is the least subscript for which $g_k(x_1, \ldots, x_{s-1})$ is a non-zero polynomial, then $g_k = f^{(1)}$.

By induction define $f^{(i+1)}(x_1, \ldots, x_{s-i-1})$ as the value of the polynomial

$$f^{(i)}(x_1, \ldots, x_{s-i})x_{s-i}^{-V_{x_{s-i}}(f^{(i)})} \quad \text{at } (x_1, \ldots, x_{s-i-1}, 0).$$

Notice that $f^{(s)}$ is a constant polynomial, i.e., $f^{(s)}$ is an element of K_0.

For any non-zero polynomial $f(x_1, \ldots, x_s)$ over K_0 define

$$W_s(f) = (V_{x_s}(f), V_{x_{s-1}}(f^{(1)}), \ldots, V_{x_1}(f^{(s-1)}), V_0(f^{(s)})).$$

We claim that W_s induces a valuation on K_s. To prove this, apply induction on s. If $s = 1$, then the assertion follows immediately from Lemma 2. Suppose as the induction hypothesis that W_{s-1} is a valuation on K_{s-1} with value group $Z^{s-1} \times G_0$ lexicographically ordered. Applying Lemma 2 with $F = K_{s-1}$, $x = x_s$ and $V^* = W_{s-1}$, we see that the mapping W defined on $K_{s-1}[x_s]$ by

$$W(g) = (V_{x_s}(g), W_{s-1}(g^*))$$

($g \neq 0$ in $K_{s-1}[x_s]$, g^* as in Lemma 2) gives a valuation on K_s with value group $Z^s \times G_0$ lexicographically ordered. Observe that for any non-zero polynomial $f(x_1, \ldots, x_s)$ over K_0, regarded as a polynomial in x_s with coefficients from the field K_{s-1}, f^* is the same as $f^{(1)}$. Consequently,

$$W(f) = (V_{x_s}(f), W_{s-1}(f^{(1)})) = W_s(f).$$

Therefore $W_s = W$ is a valuation of K_s, extending the valuation v_0 of K_0, with value group $Z^s \times G_0$ of rank $r+s$.

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