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ALMOST PERIODIC EXTENSIONS OF FUNCTIONS, II

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The following properties of a subset A of the real axis L were introduced in [1].

$\boldsymbol{I}$: A has property $\boldsymbol{I}$ (or is an *$\boldsymbol{I}$-set* or *belongs to the class* $\boldsymbol{I}$: $A \epsilon \boldsymbol{I}$) if every bounded real or complex valued function on A which is uniformly continuous on A with respect to the usual metric in L can be extended to an almost periodic (a. p.) function over L.

$\boldsymbol{I}_0$: Terms and definition are analoguous but the uniform continuity is not assumed.

As was pointed out in [1], these properties make sense for an arbitrary locally compact abelian group G instead of L. Obviously, the uniform continuity must then be understood with respect to the uniform group structure in G. No attention was given in [1] to property $\boldsymbol{I}$ except for $G = L$ (only $\boldsymbol{I}_0$ was investigated in the general case). However, we find suitable to treat here both $\boldsymbol{I}$ and $\boldsymbol{I}_0$ for arbitrary abelian locally compact groups in spite of the fact that $\boldsymbol{I}$ is irrelevant for compact groups since every set is then an $\boldsymbol{I}$-set. Similarly, two other properties defined in [1] for sequences of reals may be now reformulated for the general case:

$A \epsilon \boldsymbol{O}$ if $\mu(\tilde{A}) = 0$, $\tilde{A}$ (or $A^{\sim}$) denoting the weak closure of A, i. e. its closure in the Bohr compactification $\tilde{G}$ of the group G (for the definition see [1], p. 24) and μ being the Haar measure in $\tilde{G}$,

$A \epsilon \boldsymbol{B}$ if there is a compact set $K \subset G$ such that $A+K = G$.

The unsolved problem P 452 in [1] was: does $\boldsymbol{I}_0$ or even $\boldsymbol{I}$ imply $\boldsymbol{O}$? *The unsolved problem* P 453 *was: does* $\boldsymbol{I}_0$ *or even* $\boldsymbol{I}$ *imply the negation* $\boldsymbol{B}'$ of $\boldsymbol{B}$?

Both problems were formulated only for $G = L$ and for $A = \{a_n\}$ $(a_{n+1} > a_n)$. We must correct at this place some error in [1]: property $\boldsymbol{B}$ was there defined as $a_{n+1} - a_n < d$ $(n = 1, 2, \ldots)$. But then the implication $\boldsymbol{I} \to \boldsymbol{B}'$ is false for bounded sequences and for unbounded it follows from $\boldsymbol{I}_0 \to \boldsymbol{B}'$; thus the second question in P 453 should not have been posed at all. If we had adopted for $\boldsymbol{B}$ our present definition, $\boldsymbol{I} \to \boldsymbol{B}'$

would be equivalent with $\boldsymbol{I}_0 \to \boldsymbol{B}'$. This equivalence is true for all locally compact non-compact groups, as can be easily proved; for compact groups $\boldsymbol{B}$ is fulfilled trivially by every set and so $\boldsymbol{I} \to \boldsymbol{B}'$ is wrong. Thus, at any rate, we have to drop the second part of P 453. It remains the question: does $\boldsymbol{I}_0$ imply $\boldsymbol{B}'$ for non-compact groups? ($\mathrm{P_c}$ 453 (1)).

Now we are able to answer positively the first part of P 452 for non-discrete separable groups and $\mathrm{P_c}$ 453 for separable non-compact groups. The corresponding theorems and some corollaries resulting therefrom are object of section 2. Section 1 is devoted to a preparatory "thickening theorem" (Theorem 1). In section 3 we deal with the problem of finding a. p. extensions the Fourier series of which converge absolutely.

1. THEOREM 1. *If A is an $\boldsymbol{I}_0$-set in a separable abelian locally compact group G, then there exists a neighbourhood V of the identity (i. e. of the zero element) in G such that $E = A+V$ is an $\boldsymbol{I}$-set.*

Proof. We notice that every $\boldsymbol{I}_0$-set consists of isolated points only and so in view of the separability of G we may arrange A in a sequence $\{a_n\}$. Next we show that it is sufficient to find a compact neighbourhood V with the following property:

(*) for every 0-1 sequence $\{t_n\}$ there is an a. p. function such that $f(a_n+u) = t_n$ $(u \in V;\ n = 1, 2, \ldots)$.

If f is a real valued function, defined and uniformly continuous on E and if $a < b$, then we put $E_1 = \{x \in E: f(x) < a\}$ and $E_2 = \{x \in E: f(x) > b\}$, and we have to prove that $\tilde{E}_1 \frown \tilde{E}_2 = \emptyset$. This will show that E is an $\boldsymbol{I}$-set, in view of the evident lemma (cf. [1], Lemma 1): If f is a bounded real function defined on a subset E of a normal topological space X and the closures of the sets $\{x \in E: f(x) < a\}$ and $\{x \in E: f(x) > b\}$ are disjoint for any numbers a and $b > a$, then f has an extension to a continuous function on X.

Assume to the contrary that E_1 and E_2 have a common cluster point a in G. From (*) it follows that for every splitting $N = N_1 \smile N_2$ of the set N of positive integers the sets

$$\Big[\bigcup_{n \in N_i} (a_n+V)\Big]^{\sim} \qquad (i = 1, 2)$$

are disjoint. Hence, for each weak neighbourhood U of a there must be an index n such that there exists a point $x \in E_1 \frown (a_n+V) \frown U$ and a point $y \in E_2 \frown (a_n+V) \frown U$. Hence $x-y \in V-V$. Since V is compact, the topology in $V-V$ is the same whether this set is considered in G or in $\tilde{G}$. So for every neighbourhood W of the identity in G we can find U such that for $x, y \in U$ the condition $x-y \in V-V$ implies $x-y \in W$. Hence

(1) The subscript c stands for "correct".

for every W there would be two points $x \in E_1$ and $y \in E_2$ such that $x-y \in W$. This contradicts the uniform continuity of f on E.

Let now $\{V_n\}$ $(n = 1, 2, \dots)$ be a complete system of compact neighbourhoods of the identity in the (separable!) group G. If $t = \{t_n\}$ is a fixed 0-1 sequence, we put $N_1^t = \{n \in N: t_n = 0\}$, $N_2^t = \{n \in N: t_n = 1\}$, $A_1^t = \{a_n: n \in N_1\}$ and $A_2^t = \{a_n: n \in N_2\}$. Since $\tilde{A}_1$ and $\tilde{A}_2$ are disjoint in the (normal) space $\tilde{G}$ we can find a neighbourhood U of the identity in $\tilde{G}$ such that $\tilde{A}_1 + U$ and $\tilde{A}_2 + U$ are still disjoint. There is an n such that $V_n \subset U$ and so $(A_1 + V_n)^\sim$ and $(A_2 + V_n)^\sim$ are disjoint.

Let C be the cartesian product of $\aleph_0$ cyclic groups C_2. We consider each t as an element of C and claim that for every fixed m the set

$$Z_m = \{t: (A_1^t + V_m)^\sim \cap (A_2^t + V_m)^\sim = \emptyset\}$$

is a subgroup of C. To show this one must prove that for $t_1, t_2 \in Z_m$, $N_1 = (N_1^{t_1} \cap N_1^{t_2}) \cup (N_2^{t_1} \cap N_2^{t_2})$, $N_2 = N_2^{t_1} \dot{-} N_2^{t_2}$ ($\dot{-}$ denoting the symmetric difference) and $A_i = \{a_n: n \in N_i\}$ $(i = 1, 2)$ one has

$$(A_1 + V_m)^\sim \cap (A_2 + V_m)^\sim = \emptyset. \tag{1}$$

We observe that if a set D is disjoint with both $\tilde{D}_1 + V$ and $\tilde{D}_2 + V$, then it is also disjoint with $(D_1 \cup D_2)^\sim + V = (\tilde{D}_1 + V) \cup (\tilde{D}_2 + V)$. We put

$$D = [(A_2^{t_1} \setminus A_2^{t_2}) + V_m]^\sim, \qquad D_1 = A_1^{t_1} \cap A_1^{t_2}, \qquad D_2 = A_2^{t_1} \cap A_2^{t_2}.$$

Since $A_1 = (A_1^{t_1} \cap A_1^{t_2}) \cup (A_2^{t_1} \cap A_2^{t_2})$ and $A_2 = A_2^{t_1} \dot{-} A_2^{t_2}$ $(i = 1, 2)$ we have to prove that

(2) $D \cap (\tilde{D}_1 + V_m) = \emptyset$ and (3) $D \cap (\tilde{D}_2 + V_m) = \emptyset$.

The analogous formulae with t_1 instead of t_2 and vice versa follow then by symmetry and the four together imply (1).

The set D is contained in $(A_2^{t_1} + V_m)^\sim$ and in view of $t_1 \in Z_m$ it is disjoint with the set $(A_1^{t_1} + V_m)^\sim$ containing $\tilde{D}_1 + V_m$. So (2) holds. On the other hand, $D \subset (A_1^{t_2} + V_m)^\sim$ (since $t_2 \in Z_m$ and $A_1^t \cup A_2^t = A$) whereas $\tilde{D}_2 + V_m$ is a subset of $(A_2^{t_2} + V_m)^\sim$. Hence (3) holds in view of $t_2 \in Z_m$.

Observe that in view of the definition of Z_m the set $\{a_n + V_m\}$ fulfils partially condition (*), namely with respect to all 0-1 sequences from Z_m. We must now find V which would be good in general. We begin with proving that Z_m is measurable with respect to the Haar measure in C. For this purpose we will show that Z_m is an analytic subset of C. Let X be the space of all continuous functions on G with a topology of uniform convergence on compact sets. Since G is separable and locally compact (so σ-compact), X is a separable metrizable complete space. The almost periodic functions form a Borel set Y in X. In fact, denoting by $g = \{g_n\}$

$(g_n \epsilon G)$ an element of the (separable) space $G^{\aleph_0}$, we have

$$(f \epsilon Y) \equiv \bigwedge_{\varepsilon>0} \bigvee_{g} \bigvee_{\mu \epsilon N} \bigwedge_{z \epsilon G} \bigvee_{n \leqslant \mu} \bigwedge_{u \epsilon G} (|f(u+z)-f(u+g_n)| < \varepsilon).$$

The third and fifth quantor are obviously countable; the first can be replaced by a denumerable one (e. g. by taking ε rational), this being the case for the remaining quantors too, in view of the separability of G and $G^{\aleph_0}$ and the uniform continuity of f. The sets of the form $\{f\colon f(u_0) = a_0\}$ are evidently closed, hence Y is Borel. Now we write

$$(t \epsilon Z_m) \equiv \bigvee_{f \epsilon Y} \bigwedge_{n} \big(f(a_n + V_m) = t_n\big).$$

This is enough to see that Z_m is analytic. Since $C = \bigcup_{m=1}^{\infty} Z_m$, $Z_m \subset Z_{m+1}$ and the Z_m's are measurable, all Z_m's for m sufficiently large are of positive Haar measure and hence open subgroups of C. Since C is compact, there must be an index m_0 such that $Z_{m_0} = C$. This means that the set $\{a_n + V_{m_0}\}$ has the desired property $(*)$ and so it is an $\boldsymbol{I}$-set.

Let us remark that Theorem 1 is irrelevant for discrete groups, because the assertion is then satisfied for $V = (0)$.

2. THEOREM 2. *If G is non-discrete and separable and A is an $\boldsymbol{I}_0$-set in G, then $\mu(\tilde{A}) = 0$.*

Proof. According to Theorem 1 we choose a neighbourhood V of the identity in G so that $A+V$ be an $\boldsymbol{I}$-set. Since G is non-discrete, V consists of infinitely many points. If $x_1, x_2 \epsilon V$, $x_1 \neq x_2$, then the set $\bigcup_{i=1}^{2} (A+x_i)$ is an $\boldsymbol{I}_0$-set because every bounded function f on it can be obviously extended to a uniformly continuous function on $A+V$ and further to an a. p. function over G. In particular, one can put $f(x) = 0$ for $x \epsilon A + x_1$ and $f(x) = 1$ for $x \epsilon A + x_2$. Hence the weak closures $(A+x_1)^{\sim} = \tilde{A} + x_1$ and $(A+x_2)^{\sim} = \tilde{A} + x_2$ are disjoint. If now $x_1, x_2, x_3, \ldots$ are different points from V, then the sets $\tilde{A} + x_i$ are pairwise disjoint and the μ-measure of each of them is evidently equal to $\mu(\tilde{A})$. Since $\mu(\tilde{G}) = 1$, it must be $\mu(\tilde{A}) = 0$.

Remark 1. Theorem 2 does by no means imply that

$$\mu\Big(\big(\bigcup_{i=1}^{\infty} (A+x_i)\big)^{\sim}\Big) = 0.$$

It is still an open problem whether $\mu(\tilde{A}) = 0$ must hold for every $\boldsymbol{I}$-set in a non-compact group (**P 547**).

Remark 2. In the intrinsic language of the group G, $\mu(\tilde{A}) = 0$ for A being an $\boldsymbol{I}$-set means that for every $\varepsilon > 0$ there is a non-negative

a. p. function equal to 1 on A whose mean value is less than ε. In fact, this follows at once from the known equation

$$\text{mean value of } f = \int_{\tilde{G}} f^*(x)\,d\mu,$$

where f^* is the continuous extension of an a. p. function f over $\tilde{G}$.

THEOREM 3. *In a separable, locally compact but non-compact group G property $\boldsymbol{I}_0$ implies $\boldsymbol{B}'$.*

Proof. Let $A \subset G$ and $A \in \boldsymbol{I}_0$. By [5] (Theorem 1 ([2])) the set A has no weak accumulation points in G. Clearly, every subset $A_1 \subseteq A$ has this property too. Further, for every compact set $K \subseteq G$ we have

(i) the sets $A+K$ and A_1+K are weakly closed in G, as they are group-sums of a set weakly closed in G and a compact set.

Let us recall once more that for subsets of G the compactness and the weak compactness are equivalent, and, moreover,

(ii) on a bounded subset of G (i. e. such that its closure is compact) the strong and the weak topology coincide.

We will show now that both topologies coincide on $A+K$. For an arbitrary point $x_0 \in A+K$ we put

$$A_1 = \{y \in A\colon x_0 \notin y+K\}.$$

It is easy to see that $x_0 \in (A+K)\setminus(A_1+K)$, the set $A\setminus A_1$ is finite and, moreover, that the set $(A+K)\setminus(A_1+K)$ is bounded and weakly open in $A+K$ (see (i)). Hence, in view of (ii), the family of all sets of the form $U \cap \big((A+K)\setminus(A_1+K)\big)$, where U is open in G and $x_0 \in U$, is a complete system of neighbourhoods in $A+K$ of the point x_0 in both topologies. Consequently, if there were a compact set $K \subseteq G$ such that $A+K = G$, then the weak topology in G would be the same as the strong, hence G would be a compact group.

An obvious consequence of Theorem 3 is that an $\boldsymbol{I}_0$-sequence $\{a_n\}$ on the real line has bounded differences: $a_{n+1}-a_n < d$. However, we do not know whether an $\boldsymbol{I}$-set on the line must be of the relative measure 0, i. e. whether for an $\boldsymbol{I}$-set A there is

$$\lim_{T\to\infty} \frac{1}{T}\,|A \cap (0,T)| = 0,$$

$|\ |$ denoting the Lebesgue measure. It would be so if $\mu(\tilde{A}) = 0$.

([2]) This was proved for the real line, but the proof remains valid for every separable locally compact abelian group.

3. Let us denote by |a. p. (G)| the class of all complex valued almost periodic functions on an abelian topological group G, with absolutely convergent Fourier expansions. By the definition of the property I_0 every bounded function φ on $A \in I_0$ has an extension to an a. p. function f on G. One can ask about the existence of an extension belonging to |a. p. (G)| (3). This new property of A will be denoted by $|I_0|$. Some positive partial answers will be given below.

If $f \in$ |a. p. (G)| and $f = \sum c_n \chi_n$, where χ_n are characters of G, then we define the norm $\|f\| = \sum |c_n|$.

LEMMA 1. *If $f \in$ |a. p. (H)| and h is a homomorphism from G into an abelian topological group H, then $fh \in$ |a. p. (G)| and $\|fh\| \leqslant \|f\|$.*

This inequality can be checked directly.

LEMMA 2. *If G is compact and F_1 and F_2 are closed and disjoint subsets of G, then there is a function $f \in$ |a. p. (G)| such that $f(x) = 0$ for $x \in F_1$ and $f(x) = 1$ for $x \in F_2$.*

Proof. From the well-known theorem of Pontrjagin it follows that there exists a homomorphism h of G into a finitely dimensional torus H such that the sets $h(F_1)$ and $h(F_2)$ are disjoint. There is a separating function g, i. e.

$$g(y) = \begin{cases} 0 & \text{for } y \in h(F_1), \\ 1 & \text{for } y \in h(F_2), \end{cases} \tag{4}$$

which is regular enough (say indefinitely derivable) to have $g \in$ |a. p. (H)|. Lemma 1, the property (4) and the evident inclusions

$$F_1 \subseteq h^{-1}h(F_1) \quad \text{and} \quad F_2 \subseteq h^{-1}h(F_2)$$

imply that the function $f = gh$ has all required properties.

THEOREM 4. *If $A \subset G$ and $A \in I_0$, and φ is a function with a finite range, then φ admits an extension to an f from* |a. p. (G)|.

Proof. It suffices to consider a function φ taking the values 0 and 1 only. Let k denote the natural embedding of G into its Bohr compactification $\tilde{G}$. From the property I_0 of A follows that the sets

$$\{k(x): x \in A \text{ and } \varphi(x) = 0\} \quad \text{and} \quad \{k(x): x \in A \text{ and } \varphi(x) = 1\}$$

have disjoint closures F_1, F_2 in $\tilde{G}$. By Lemma 2 there exists a function $g \in$ |a. p. (G)| which is equal to 0 on F_1 and is equal to 1 on F_2. Obviously the function $f = gk$ has the desired property.

(3) This question, in the case of real line, has been formulated by J.-P. Kahane, who has also observed that an I_0-set A has property $|I_0|$ if and only if $\tilde{A}$ is the Helson set (see [2], p. 139) in the Bohr compactification.

For some special class of $\boldsymbol{I}_0$-sets on the real line L we are able to give a complete solution.

THEOREM 5. *Every sequence of reals* $\{t_n\}$ $(n = 1, 2, \ldots)$ *such that*

$$t_1 > 0, \quad t_{n+1}/t_n > 1+\delta \quad (\delta > 0) \tag{5}$$

belongs to $|\boldsymbol{I}_0|$.

Proof. Strzelecki has shown in [7] that sequences satisfying (5) belong to $\boldsymbol{I}_0$. Now, let us observe that if A is an arbitrary $\boldsymbol{I}_0$-set and if it can be split into a finite number of parts $A_1, \ldots, A_k$ such that $A_i \epsilon |\boldsymbol{I}_0|$ $(i = 1, \ldots, k)$, then $A \epsilon |\boldsymbol{I}_0|$. In fact, by Theorem 4 there are functions $e_1, \ldots, e_k \epsilon |\text{a. p.}|$ such that e_i is equal to 1 on A_i and vanishes on the remaining A_j. If φ is a bounded function on A and f_i denotes an $|\text{a. p.}|$ extension of $\varphi|_{A_i}$, then it is easy to check that $\sum_{i=1}^{k} e_i f_i$ is an $|\boldsymbol{I}_0|$-extension of φ.

In view of this remark we can restrict our consideration to sequences satisfying

$$t_1 > 0, \quad t_{n+1}/t_n > 3+\delta \quad (\delta > 0). \tag{6}$$

Mycielski has proved [4] that, given a sequence $A = \{t_n\}$ fulfilling (6), there is a periodic continuous function $F(t)$ such that every 0-1 function on A has an extension on the whole line of the form $F(qt)$, where q is a suitably chosen constant $\neq 0$. If we take $F(t)$ sufficiently smooth, e. g. continuously derivable, then F is an $|\text{a. p.}|$-function and all functions $F(qt)$ have clearly the same norm:

$$\|F(q\cdot)\| = C \text{ for all } q \neq 0. \tag{7}$$

Let φ be a bounded function on A. We may assume that $0 \leqslant \varphi \leqslant 1$. We form now the dyadic expansion of φ:

$$\varphi = \sum_{i=1}^{\infty} j_i/2^i, \quad \text{where} \quad j_i = 0 \text{ or } 1,\ i = 1, 2, \ldots$$

We know that there is an $|\text{a. p.}|$-extension f_i of j_i such that $\|f_i\| = C$ (see (7)), hence the series $\sum_{i=1}^{\infty} f_i/2^i$ (which is absolutely convergent in the norm $\|\cdot\|$) gives the required extension of φ.

The general problem of the equality $\boldsymbol{I}_0 = |\boldsymbol{I}_0|$ remains open (**P 548**). Let us mention that the answer would be positive if we were able to prove the following

CONJECTURE. *If T is a linear operator from a Banach space X into the space $m(A)$ of all bounded numerical functions defined on an abstract set A, such that the range of T contains the set $E(A)$ of all* 0-1 *valued functions on A, then $T(X) = m(A)$.*

In order to see that the Conjecture really implies $\boldsymbol{I}_0 = |\boldsymbol{I}_0|$ one has to put $X = |\mathrm{a.\,p.}\,(G)|$ and $T(f) = f|_A$ $(f \in X, A \in \boldsymbol{I}_0)$ and to apply Theorem 4.

On the other hand, it turns out that the Conjecture formulated above is equivalent to the following (unpublished) problem of S. Mazur and W. Orlicz: A set Q in a Banach space is called *barrel* if every sequence of linear functionals $\xi_1, \xi_2, \ldots$ which is pointwise convergent to zero on Q is a bounded sequence, i. e. $\sup_n \|\xi_n\| < \infty$. Is the set $E(A)$ a barrel in $m(A)$?

Added in proof. Problems P 452 and P 453 have already been solved (see [6] and [3] respectively).

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