COLLOQUIUM MATHEMATICUM

VOL. XXIII 1971 FASC. 1

REMARKS ON BAIRE THEOREM FOR H-CLOSED SPACES

BY

J. MIODUSZEWSKI (KATOWICE)

The following fact is very familiar: a compact Hausdorff space is of the cardinality of at least continuum whenever it is dense in itself (Alexandroff and Urysohn [1]). Recently some theorems have been proved (Körber [6], Veličko [8] and also Bourbaki [2], exercise 24a in chapter I, § 9) which, roughly speaking, assert that an H-closed space is uncountable whenever it is dense in itself, or, in a more strong form, that a countable H-closed space does not contain any subset dense in itself.

These theorems on H-closed spaces follow from the following more general

THEOREM. *An H-closed space X which is a countable union of compact subspaces $A_1, A_2, \ldots$ does not contain any subset Y such that $Y \cap A_1$, $Y \cap A_2, \ldots$ are all compact and nowhere dense in Y.*

Before the proof let us notice corollaries and a question.

If $A_1, A_2, \ldots$ are all one-point subspaces, we get the results formulated above. Another corollary (for $Y = X$) is a kind of Baire category theorem: an H-closed space is not a countable union of compact nowhere dense subspaces. This implies immediately that a Hausdorff space which is a countable union of compact nowhere dense subspaces admits no contraction of its topology to an H-closed topology and, in consequence, to a minimal Hausdorff topology (in fact, after a contraction, compact and nowhere dense subspaces remain compact and nowhere dense). This result was obtained by Holsztyński [5] (see Herrlich [3] for a special case of the set of all rational numbers) under an additional hypothesis which occurs therefore superfluous.

Herrlich [4] shows that the Baire theorem in the form "a space X is not a countable union of clased nowhere dense subspaces" is for minimal Hausdorff spaces X not true. A counter-example may be described as follows.

Let X be a compact Hausdorff space with a countable, dense and nowhere dense subspace D. Let us expand the topology on X by adding D

as a new open subset. The topology remains H-closed (see Mioduszewski and Rudolf [7], chapter I, § 3, for more general expansions preserving H-closedness of topologies). Now let us stick together two copies of the space just constructed, $X\times\{0\}$ and $X\times\{1\}$, along the closed subset $X-D$; more precisely, form a pushout diagram

$$\begin{array}{ccc} X-D & \longrightarrow & X\times\{0\} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle q|X\times\{0\}} \\ X\times\{1\} & \xrightarrow[q|X\times\{1\}]{} & Z \end{array}$$

where i and j are embeddings given by $i(x) = (x, 0)$ and $j(x) = (x, 1)$ for $x\in X-D$ and Z is a quotient space of $X\times\{0\}\cup X\times\{1\}$ given by a quotient map q identifying points $(x, 0)$ and $(x, 1)$ for $x\in X-D$. Since sticking together of two copies of a Hausdorff space along a closed subset leads to a Hausdorff space, the space Z is Hausdorff. As an image of H-closed space $X\times\{0\}\cup X\times\{1\}$, the space Z is H-closed. The space Z is semiregular. In fact, a regular open base is the family consisting 1° of all regularly open (in D) subsets of both copies, $D\times\{0\}$ and $D\times\{1\}$, of D, and 2° of all "symmetric" regularly open neighbourhoods of points going from $X-D$, i.e. of all sets of the form $q(U\times\{0\}\cup U\times\{1\})$, where U is regularly open in X. Thus Z is minimal Hausdorff. The sets $N_x = \{(x, 0), (x, 1)\}\cup \mathrm{Im}(X-D)$ are closed and nowhere dense in Z and $Z = \bigcup\{N_x\colon x\in D\}$ (the symbol Im denotes the image under $q\circ i$ $(= q\circ j)$). This means that Z is a countable union of closed nowhere dense subspaces.

We shall also give a counter-example to Baire theorem for H-closed spaces which admit contractions to compact Hausdorff spaces. To do this, let us take a "half" of the preceding example, namely the space X with the expanded, by adding D, topology. The sets $M_x = \{x\}\cup(X-D)$ are closed and nowhere dense in X and $X = \cup\{M_x : x\in D\}$.

In both examples the closed nowhere dense subspaces are not H-closed. In fact, M_x differs from $X-D$ in the isolated in M_x point x only, and N_x differs from $\mathrm{Im}(X-D)$ in the isolated in N_x two points $(x, 0)$ and $(x, 1)$. But $X-D$ and $\mathrm{Im}(X-D)$ are both homeomorphic with $X-D$ in its original topology induced from the compact space X, and $X-D$ is not H-closed because D is dense and nowhere dense in the original topology of X.

On the other hand, according to the theorem, these closed nowhere dense subspaces cannot be compact. Thus the following question seems to be open:

QUESTION. *Is Baire theorem true for H-closed spaces with respect to H-closed subspaces?* (**P 715**)

Proof of the theorem. Let $x_1 \in Y - A_1$. For each $x \in A_1$ let U_x be an open neighbourhood of x in X such that $x_1 \notin \mathrm{Cl}\, U_x$. Since A_1 is compact, there exists a finite subcovering P_1 of the covering $\{U_x : x \in A_1\}$ of A_1. We have $x_1 \notin \mathrm{Cl} \cup P_1$. Let us denote $\mathrm{Cl} \cup P_1$ by B_1.

Let us proceed by induction. Suppose that there are defined points $x_1, \ldots, x_k$ of Y and finite and open in X coverings P_1 of $A_1, \ldots, P_k$ of A_k such that $x_i \notin B_1 \cup \ldots \cup B_i$ for $i = 1, \ldots, k$, where $B_j = \mathrm{Cl} \cup P_j$. Define x_{k+1} to be a point of Y such that $x_{k+1} \notin B_1 \cup \ldots \cup B_k \cup A_{k+1}$. Such a point exists, since $Y - (B_1 \cup \ldots \cup B_k)$ is an open and non-empty subset of Y and $Y \cap A_{k+1}$ is nowhere dense in Y. For each $x \in A_{k+1}$ let U_x be an open neighbourhood of x in X such that $x_{k+1} \notin \mathrm{Cl}\, U_x$. Since A_{k+1} is compact, there exists a finite subcovering P_{k+1} of the covering $\{U_x : x \in A_{k+1}\}$ of A_{k+1}. Let $B_{k+1} = \mathrm{Cl} \cup P_{k+1}$. We have $x_{k+1} \notin B_{k+1}$, hence $x_{k+1} \notin B_1 \cup \ldots \cup B_k \cup B_{k+1}$.

Thus there is defined open covering $P_1 \cup P_2 \cup \ldots$ of X such that for no finite subfamily Q of it there is $X = \mathrm{Cl} \cup Q$. A contradiction.

REFERENCES

[1] P. S. Alexandroff et P. S. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verhandelingen der Koninklijke Akademie van Wetenschappen, Amsterdam, Deel XIV, Nr 1 (1929), p. 1-96.

[2] N. Bourbaki, *Topologie générale*, Troisième édition entièrement refondue, Paris 1960.

[3] H. Herrlich, *T_ν-Abgeschlossenheit und T_ν-Minimalität*, Mathematische Zeitschrift 88 (1965), p. 285 - 294.

[4] — *Nicht alle T_2-minimalen Räume sind von 2-Kategorie*, ibidem 91 (1966), p. 185.

[5] W. Holsztyński, *О минимальных хаусдорфовых пространствах и T_1-бикомпактах*, Доклады Академии Наук СССР 178 (1968), p. 24 - 26.

[6] K. H. Körber, *Eine Bemerkung über Bairesche Kategorie der a.-abgeschlossenen Räume*, Mathematische Nachrichten 38 (1968), p. 67 - 71.

[7] J. Mioduszewski and L. Rudolf, *H-closed and extremally disconnected spaces*, Dissertationes Mathematicae 66 (1969).

[8] Н. Величко, *О H-замкнутых топологических пространствах*, Сибирский Математический Журнал 8 (1967), p. 754 - 763.

SILESIAN UNIVERSITY, KATOWICE

Reçu par la Rédaction le 29. 12. 1969