COLLOQUIUM MATHEMATICUM

VOL. XXX 1974 FASC. 1

A CHARACTERIZATION OF GAUSSIAN PROCESSES THAT ARE MARKOVIAN

BY

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In this paper we deal with real stochastic processes $\xi(t)$, defined on an interval T, with the expected value $E\xi(t) = 0$ and satisfying the following condition:

$$E(\xi(t_n) \mid \xi(t_1), \ldots, \xi(t_{n-1})) = E(\xi(t_n) \mid \xi(t_{n-1}))$$

for every sequence $t_1 < t_2 < \ldots < t_n$, $t_i \in T$.

Such processes $\xi(t)$ and sequences $\{\xi(t_i)\}_{i=1,2,\ldots,n}$ are said *to have Markov property*.

A process $\xi(t)$ is called *Markovian* if, for an arbitrary sequence $t_1 < t_2 < \ldots < t_n$,

$$P(\xi(t_n) \in A \mid \xi(t_1), \ldots, \xi(t_{n-1})) = P(\xi(t_n) \in A \mid \xi(t_{n-1})).$$

If every random vector $(\xi(t_1), \ldots, \xi(t_n))$ has Gaussian distribution, then $\xi(t)$ is said to be a *Gaussian process*.

The aim of this paper is to give necessary and sufficient conditions for the function $f(u, v) = E\xi(u)\xi(v)$ to be the covariance function of a Gaussian Markov process.

LEMMA 1. *A Gaussian process $\xi(t)$ is Markovian if and only if it has Markov property* (1).

LEMMA 2. *The conditional expected value $E(\xi(t_n) \mid \xi(t_1), \ldots, \xi(t_{n-1}))$ is the orthogonal projection of the random variable $\xi(t_n)$ onto a subspace of the Hilbert space L^2 which is generated by random variables $\xi(t_1), \ldots, \xi(t_{n-1})$* (op. cit. 33.3.A).

Now, let $\xi(t)$ be a Gaussian process. The matrix of covariance of the random vector $(\xi(t_1), \ldots, \xi(t_n))$ will be denoted by $\Lambda(t_1, \ldots, t_n)$. To simplify the notation, let us write ξ_i instead of $\xi(t_i)$. We shall use the following lemma:

(1) For the proof see 38.1.A, in M. Loève, *Probability theory*, D. van Nostrand Company Inc.

LEMMA 3. *If* $\det \Lambda(t_1, \ldots, t_n) > 0$, *then the random variables* $\xi_1, \xi_2, \ldots, \xi_n$ *are linearly independent.*

At the beginning let us restrict our attention to the case where the assumption of Lemma 3 is satisfied for arbitrary numbers $t_1 < t_2 < t_3$. The random variables ξ_1 and ξ_3 can be presented in the forms $\xi_1 = a\xi_2 + \eta$ and $\xi_3 = b\xi_2 + \psi$, where a and b are chosen so that $E\eta\xi_2 = E\psi\xi_2 = 0$. It follows from Lemma 3 that $\eta, \psi \not\equiv 0$.

THEOREM 1. *The following conditions are equivalent:*

(i) *the random variables* ξ_1, ξ_2 *and* ξ_3 *have Markov property;*

(ii) $E\eta\psi = 0$;

(iii) $\lambda_{13}\lambda_{22} = \lambda_{12}\lambda_{23}$, *where* $\lambda_{ij} = E\xi_i\xi_j$.

Proof. We can present the random variable ξ_3 in the following two ways: $\xi_3 = a\xi_1 + \beta\xi_2 + \gamma$, where $\gamma \not\equiv 0$, $E\xi_1\gamma = E\xi_2\gamma = 0$; and $\xi_3 = b\xi_2 + \psi$. Hence $E(\xi_3 | \xi_1, \xi_2) = a\xi_1 + \beta\xi_2$ and $E(\xi_3 | \xi_2) = b\xi_2$.

It is obvious that (i) $\Leftrightarrow a\xi_1 + \beta\xi_2 = b\xi_2 \Leftrightarrow a\xi_1 + (\beta - b)\xi_2 = 0 \Leftrightarrow$ $\Leftrightarrow a = 0$ and $b = \beta$ (since ξ_1 and ξ_2 are linearly independent) $\Leftrightarrow \beta = b$ and $\gamma = \psi \Leftrightarrow E\eta\xi_1 = 0 \Leftrightarrow E\eta\psi = 0 \Leftrightarrow$ (ii).

Next, we compute the coefficients a and b:

$$E(\xi_1 - a\xi_2)\xi_2 = 0 \Leftrightarrow a = \frac{\lambda_{12}}{\lambda_{22}}, \qquad E(\xi_3 - b\xi_2)\xi_2 = 0 \Leftrightarrow b = \frac{\lambda_{23}}{\lambda_{22}}.$$

So condition (ii) is equivalent to the condition

$$E\left(\xi_1 - \frac{\lambda_{12}}{\lambda_{22}}\xi_2\right)\left(\xi_3 - \frac{\lambda_{23}}{\lambda_{22}}\xi_2\right) = 0 \Leftrightarrow \lambda_{13}\lambda_{22} = \lambda_{12}\lambda_{23}.$$

Thus, the question arises whether $\xi(t)$ is a Markov process if, for arbitrary $t_1 < t_2 < t_3$, condition (i) holds. The positive answer follows from the fact that, under the above assumption, the random variables $\xi_1, \xi_2, \ldots, \xi_k, \xi_{k+1}$ have Markov property for an arbitrary sequence $t_1 < t_2 < \ldots < t_k < t_{k+1}$, and that is what the following theorem states:

THEOREM 2. *Let* $(\xi_1, \xi_2, \ldots, \xi_k, \xi_{k+1})$ *be a Gaussian vector and let every triple of random variables* ξ_i, ξ_k, ξ_{k+1} $(i = 1, 2, \ldots, k-1)$ *have Markov property. Then the random variables* $\xi_1, \xi_2, \ldots, \xi_k, \xi_{k+1}$ *also have Markov property.*

Proof. Let

$$\text{(1)} \qquad \xi_1 = a_1\xi_k + \eta_1,\ \xi_2 = a_2\xi_k + \eta_2,\ \ldots,\ \xi_{k-1} = a_{k-1}\xi_k + \eta_{k-1},$$
$$\xi_{k+1} = a_{k+1}\xi_k + \eta_{k+1}, \quad \text{where } \eta_i = 0 \text{ and } E\eta_i\xi_k = 0.$$

Simultaneously,

$$E(\xi_{k+1} | \xi_1, \xi_2, \ldots, \xi_k) = a_1\xi_1 + a_2\xi_2 + \ldots + a_k\xi_k$$

implies

(2) $$\xi_{k+1} = a_1\xi_1 + a_2\xi_2 + \ldots + a_k\xi_k + \delta,$$
$$\text{where } \mathrm{E}\delta\xi_i = 0 \text{ for } i = 1, 2, \ldots, k.$$

Moreover, $\mathrm{E}(\xi_{k+1}|\xi_k) = a_k'\xi_k$ implies $\xi_{k+1} = a_k'\xi_k + \delta'$ and $\mathrm{E}\delta'\xi_k = 0$. To prove the theorem it suffices to show that $a_1 = a_2 = \ldots = a_{k-1} = 0$. Substituting (1) to (2), we get

$$(a_1\alpha_1 + a_2\alpha_2 + \ldots + a_{k-1}\alpha_{k-1})\xi_k + \eta_1 a_1 + \eta_2 a_2 + \ldots + \eta_{k-1}a_{k-1} + \delta$$
$$= \alpha_{k+1}\xi_k + \eta_{k+1}.$$

Since $\xi_1, \xi_2, \ldots, \xi_k$ are linearly independent and since

$$b_1\eta_1 + \ldots + b_{k-1}\eta_{k-1} + b_k\xi_k$$
$$= b_1\xi_1 + \ldots + b_{k-1}\xi_{k-1} + (b_k - \alpha_1 b_1 - \ldots - \alpha_{k-1}b_{k-1})\xi_k = 0$$

implies

$$b_1 = b_2 = \ldots = b_{k-1} = b_k = 0,$$

we infer that the variables $\eta_1, \eta_2, \ldots, \eta_{k-1}, \xi_k$ are also linearly independent.

Write

$$\nu = (a_1\alpha_1 + a_2\alpha_2 + \ldots + a_{k-1}\alpha_{k-1} + a_k)\xi_k + \eta_1 a_1 + \ldots + \eta_{k-1}a_{k-1}.$$

The assumption and the fact that the random variables ξ_i, ξ_k, ξ_{k+1} have Markov property result in equalities $\mathrm{E}\eta_{k+1}\eta_i = 0$ for $i = 1, 2, \ldots, k-1$, and $\mathrm{E}\eta_{k+1}\xi_k = 0$. Simultaneously,

$$\nu - \alpha_{k+1}\xi_k = \eta_{k+1} - \delta \quad \text{and} \quad \mathrm{E}\eta_{k+1}(\nu - \alpha_{k+1}\xi_k) = 0$$

as well as

$$\mathrm{E}\delta(\nu - \alpha_{k+1}\xi_k) = 0.$$

So $\mathrm{E}(\nu - \alpha_{k+1}\xi_k)(\eta_{k+1} - \delta) = 0$ and we infer that $\eta_{k+1} = \delta$, and $\nu = \alpha_{k+1}\xi_k$ implies $a_1 = a_2 = \ldots = a_{k-1} = 0$ and $a_k = \alpha_{k+1}$.

Now we drop the assumption that $\det \Lambda(t_1, t_2, \ldots, t_n) > 0$ but preserve the condition $\mathrm{E}\xi^2(t) > 0$ for every t. In the proof of Theorem 1 we used the linear independence of random variables ξ_1 and ξ_2 only. If this is not the case, i.e., if $\xi_2 = k\xi_1$, then, on one hand, the sequence ξ_1, ξ_2, ξ_3 has Markov property and, on the other hand, $\lambda_{12} = k\lambda_{11}$, $\lambda_{22} = k^2\lambda_{11}$, $\lambda_{23} = k\lambda_{13}$, and the equality $\lambda_{11}\lambda_{13} = \lambda_{12}\lambda_{23}$ is fulfilled.

Similarly, in the proof of Theorem 2 we used the linear independence of random variables $\xi_1, \xi_2, \ldots, \xi_k$ only. If this does not hold, then a basis of the linear space they generate can be chosen so that it contains the random variable ξ_k. Let $\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_l}, \xi_k$ be such a basis. Then

$$\mathrm{E}(\xi_{k+1}|\xi_k, \xi_{k-1}, \ldots, \xi_1) = \mathrm{E}(\xi_{k+1}|\xi_k, \xi_{i_1}, \ldots, \xi_{i_l})$$

and, using Theorem 2, we get

$$\mathrm{E}(\xi_{k+1}|\xi_k, \xi_{i_1}, \ldots, \xi_{i_l}) = \mathrm{E}(\xi_{k+1}|\xi_k).$$

Putting together Theorems 1 and 2 and taking into consideration the above remarks, we can formulate the following

THEOREM 3. *Let $\xi(t)$ be a Gaussian process with expected value zero and such that $\mathrm{E}\xi^2(t) > 0$ for every t. Then it is a Markov process if and only if $\lambda_{13}\lambda_{22} = \lambda_{12}\lambda_{23}$ for arbitrary $t_1 < t_2 < t_3$.*

If the notation $f(u,t) = \mathrm{E}\xi(u)\xi(t)$ for the covariance function is introduced, then we may express condition (ii) of Theorem 1 by means of the functional equation

$$f(u,t)f(v,v) = f(u,v)f(v,t), \quad \text{where } u \leqslant v \leqslant t. \tag{3}$$

Now we deal with solving this equation in the class of functions satisfying the condition $f(t,u) = f(u,t)$. First, we restrict our attention to the case $f(u,t) \neq 0$ for arbitrary $u, t \in T$.

Take an arbitrary point $u_0 \in T$. Then two numbers $g(u_0)$ and $h(u_0)$ can be chosen so that $f(u_0, u_0) = g(u_0)h(u_0)$. Let

$$\frac{f(u,u_0)}{g(u_0)} = h(u) \text{ for } u \geqslant u_0 \quad \text{and} \quad \frac{f(u,u_0)}{h(u_0)} = g(u) \text{ for } u \leqslant u_0.$$

After transformation we obtain

$$g(u) = \frac{f(u,u_0)g(u_0)}{f(u_0,u_0)} \quad \text{for } u \leqslant u_0$$

and

$$h(u) = \frac{f(u,u_0)h(u_0)}{f(u_0,u_0)} \quad \text{for } u \geqslant u_0.$$

Let, at the same time,

$$\frac{f(u,u)}{h(u)} = g(u) \text{ for } u \leqslant u_0 \quad \text{and} \quad \frac{f(u,u)}{g(u)} = h(u) \text{ for } u \geqslant u_0.$$

Let us examine dependence of solutions of the considered functional equation on the functions g and h. Consider the following cases:

1. $u \leqslant u_0 \leqslant v$. It follows that

$$f(u,v)f(u_0,u_0) = f(u,u_0)f(v,u_0)$$

implies

$$f(u,v) = \frac{g(u)h(u_0)g(u_0)h(v)}{h(u_0)g(u_0)} = g(u)h(v) = g\big(\min(u,v)\big)h\big(\max(u,v)\big).$$

2. $u \leqslant v \leqslant u_0$. It follows that

$$f(u, u_0)f(v, v) = f(u, v)f(v, u_0)$$

implies

$$f(u, v) = \frac{f(u, u_0)f(v, v)}{f(v, u_0)} = \frac{h(u_0)g(u)g(v)h(v)}{g(v)h(u_0)} = g(u)h(v)$$
$$= g\big(\min(u, v)\big)h\big(\max(u, v)\big).$$

3. $u_0 \leqslant u \leqslant v$. It follows that

$$f(u_0, v)f(u, u) = f(u_0, u)f(u, v)$$

implies

$$f(u, v) = \frac{f(u_0, v)f(u, u)}{f(u_0, u)} = \frac{g(u_0)h(v)g(u)h(u)}{g(u_0)h(u)} = g(u)h(v)$$
$$= g\big(\min(u, v)\big)h\big(\max(u, v)\big).$$

So every solution is of the form

$$f(u, v) = g\big(\min(u, v)\big)h\big(\max(u, v)\big).$$

It is easy to check that every function of this form is a solution of the considered functional equation. Thus we proved

THEOREM 4. *Solutions of equation* (3) *satisfying conditions* $f(u, v) = f(v, u) \neq 0$ *for arbitrary* $u, v \in T$ *are functions of the form*

$$f(u, v) = g\big(\min(u, v)\big)h\big(\max(u, v)\big).$$

Remark. The condition $f(u, v) \neq 0$ is essential. Namely, for $u < v$, it follows that $f(u, v) = g(u)h(v) = 0$. Then $f(u, u) = g(u)h(u) = 0$ and $f(v, v) = g(v)h(v) = 0$ which contradicts the assumption $f(u, u) \neq 0$ for every $u \in T$.

Now we examine more precisely the set of the points (u, v) for which $f(u, v) = 0$ and f is a solution of equation (3). We prove

THEOREM 5. *The set of zeros of a solution of equation* (3) *is of the form*

$$\begin{aligned} A &= \{(u, t)\colon f(u, t) = 0,\ u, t \in T\} \\ &= \bigcup_{\theta \in \Xi_1} \{(u, t)\colon \min(u, t) < \theta;\ \max(u, t) > \theta;\ u, t \in T\} \cup \\ &\cup \bigcup_{\theta \in \Xi_2} \{(u, t)\colon \min(u, t) \leqslant \theta;\ \max(u, t) > \theta;\ u, t \in T\} \cup \\ &\cup \bigcup_{\theta \in \Xi_3} \{(u, t)\colon \min(u, t) < \theta;\ \max(u, t) \geqslant \theta;\ u, t \in T\}, \end{aligned}$$

where Ξ_1, Ξ_2, Ξ_3 *are certain disjoint sets in* T.

Proof. The proof will be conducted in four steps.

(a) Let $f(u_0, t_0) = 0$, where $u_0 < t_0$. From equation (3) we have

$$f(u,t) = \frac{f(u,u_0)f(t_0,t)}{f(t_0,t_0)} \quad \text{for } u \leqslant t_0 \leqslant t.$$

Since $f(u_0, t_0) = 0$, we have

$$f(u_0,t) = \frac{f(u_0,t_0)f(t_0,t)}{f(t_0,t_0)} = 0 \quad \text{for } t \geqslant t_0.$$

Similarly,

$$f(u,t) = \frac{f(u,u_0)f(u_0,t)}{f(u_0,u_0)} \quad \text{for } u \leqslant u_0 \leqslant t,$$

whence

$$f(u,t_0) = \frac{f(u,u_0)f(u_0,t_0)}{f(u_0,u_0)} = 0 \quad \text{for } u \leqslant u_0.$$

Next, take an arbitrary $u_1 \leqslant u_0$. Since $f(u_1, t_0) = 0$, applying the former arguments, we obtain $f(u_1, t) = 0$ for $t \geqslant t_0$ and $u_1 \leqslant u$. Finally, we get $f(u, t) = 0$ for $\min(u, t) \leqslant u_0$ and $\max(u, t) \geqslant t_0$.

(b) Take an arbitrary $u_0 \leqslant v_0 \leqslant t_0$. Then

$$f(u,t) = \frac{f(u,v_0)f(v_0,t)}{f(v_0,v_0)} \quad \text{for } u \leqslant v_0 \leqslant t.$$

We have

$$f(u_0,t_0) = 0 = \frac{f(u_0,v_0)f(v_0,t_0)}{f(v_0,v_0)}$$

and, therefore,

$$f(u_0,v_0) = 0 \quad \text{or} \quad f(v_0,t_0) = 0.$$

If the first case holds, then, using (a), we have $f(u, t) = 0$ for $\min(u, t) \leqslant u_0$ and $\max(u, t) \geqslant v_0$. Analogously, in the other case, $f(u, t) = 0$ for $\min(u, t) \leqslant v_0$ and $\max(u, t) \geqslant t_0$.

(c) Let $v_1 = (u_0 + t_0)/2$. It follows from (b) that at least one of two possibilities holds:

I. $f(u, t) = 0$ if $\min(u, t) \leqslant u_0$ and $\max(u, t) \geqslant v_1$.

II. $f(u, t) = 0$ if $\min(u, t) \leqslant v_1$ and $\max(u, t) \geqslant t_0$.

We iterate our procedure putting $v_2 = (u_0 + v_1)/2$ in case I and $v_2 = (v_1 + t_0)/2$ in case II. We obtain a sequence v_n converging to a certain $\bar{v}$ which has the following property: $f(u, t) = 0$ if $\min(u, t) < \bar{v}$ and $\max(u, t) > \bar{v}$.

(d) We do not know yet what occurs when $\min(u, t) = \bar{v}$ or $\max(u, t) = \bar{v}$. We know that

$$f(u, t) = \frac{f(u, \bar{v})f(\bar{v}, t)}{f(\bar{v}, \bar{v})} = 0 \quad \text{for } u < \bar{v} < t.$$

Hence $f(u, \bar{v}) = 0$ or $f(\bar{v}, t) = 0$. Let $u_n < \bar{v}$ and $\bar{v} < t_n$; moreover, $u_n \uparrow \bar{v}$ and $t_n \downarrow \bar{v}$. For an arbitrary n, $f(u_n, \bar{v}) = 0$ or $f(\bar{v}, t_n) = 0$. It follows from (a) that if $f(u_{n_0}, \bar{v}) = 0$, then also $f(u_n, \bar{v}) = 0$ for $u_n \leqslant u_{n_0}$, and if $f(t_{n_0}, \bar{v}) = 0$, then $f(t_n, \bar{v}) = 0$ for $t_n \geqslant t_{n_0}$. So, for at least one of these sequences, the function f must take the value zero, i.e., $f(u_n, \bar{v}) = 0$ or $f(t_n, \bar{v}) = 0$ for an arbitrary n. In other words, $f(u, t) = 0$ if $\min(u, t) = \bar{v}$ or $f(u, t) = 0$ if $\max(u, t) = \bar{v}$. Suppose that $f(u, t) = 0$ if $\min(u, t) = \bar{v}$. There arises the question what occurs if $\max(u, t) = \bar{v}$. Take

$$\sup_u \{u: f(u, \bar{v}) = 0 \text{ and } u < \bar{v}\} = \tilde{u}.$$

We infer from (a) that $f(u, \bar{v}) = 0$ for $u < \tilde{u}$. Let $u'' < u' < \bar{v}$ and $f(u'', \bar{v}) = 0, f(u', \bar{v}) = 0$. From equation (3) we infer that $f(u'', \bar{v})f(u', u') = f(u'', u')f(u', \bar{v})$ implies $f(u', u'') = 0$. By (c), there exist a $\bar{v}'$ such that $u'' \leqslant \bar{v}' \leqslant u'$ and $f(u, t) = 0$ for $\min(u, t) < \bar{v}'$ and $\max(u, t) > \bar{v}'$. Every point $(u, \bar{v})$ with $u < u''$ satisfies these conditions, therefore, it can be treated as an element belonging to

$$\{(u, v): f(u, v) = 0, \max(u, v) > \bar{v}', \min(u, t) < \bar{v}'\}.$$

u'' is arbitrarily close to $\tilde{u}$, so it is true for all $u < \tilde{u}$. Similarly, if $f(\bar{v}, \tilde{u}) = 0$, then, arguing in the same way, we obtain $f(u, \tilde{u}) = 0$ for $u > \tilde{u}$, i.e., $\tilde{u}$ can be treated as $\bar{v}'$.

We have finally shown that every point (u, t), for which $f(u, t) = 0$, can be treated as an element of one of sets defined in the assertion of the theorem and that each of these sets lies in the set of zeros of the function f.

COROLLARY 1. *If* $f(u_0, t_0) = 0$, *where* $u_0 < t_0$, *and* $f(u, t)$ *is a solution of equation* (3), *then there exists a* $\bar{v}$, $u_0 \leqslant \bar{v} \leqslant t_0$, *with the following property*: $f(u, t) = 0$ *if* $\min(u, t) < \bar{v}$ *and* $\max(u, t) > \bar{v}$.

The proof follows immediately from (c) of the proof of Theorem 5.

COROLLARY 2. *For a continuous Gaussian Markov process, the condition* $f(t, t) > 0$ *for every* $t \in T$ *implies the condition* $f(u, t) \neq 0$ *for every* $u, t \in T$.

Proof. Let $f(u_0, t_0) = 0$ for certain $u_0, t_0 \in T$ such that $u_0 < t_0$. Theorem 5 implies the existence of a $\bar{v}$, $u_0 \leqslant \bar{v} \leqslant t_0$, such that if $u < \bar{v} < t$, then $f(u, t) = 0$. Let $u_n \uparrow u_0$ and $t_n \downarrow t_0$. Then $0 = f(u_n, t_n) \to f(\bar{v}, \bar{v}) = 0$.

Now we can determine the form of solutions of equation (3) without assuming $f(u, t) \neq 0$ for $u \neq t$.

THEOREM 6. *Solutions of equation* (3) *satisfying conditions* $f(u, t) = f(t, u)$ *and* $f(u, u) > 0$ *for* $u \in T$ *are functions of the form*

$$f(u, t) = \begin{cases} g(\min(u, t))h(\max(u, t)) & \text{for } (u, t) \notin A, \\ 0 & \text{for } (u, t) \in A, \end{cases}$$

where A *is the set defined in the assertion of Theorem* 5.

A proof of this theorem will be preceded by the following lemma:

LEMMA 4. *If* Ξ_i *are sets defined in the assertion of Theorem* 5, *then the set* $\Xi = \Xi_1 \cup \Xi_2 \cup \Xi_3$ *is closed.*

Proof. Let $\bar{v} = \lim_n \theta_n$ and $\theta_n \in \Xi$. We can take a subsequence $\theta_{n'}$ of the sequence θ_n so that, say, $\theta_{n'} \leqslant \bar{v}$. If $\min(u, t) < \theta_{n'}$ and $\max(u, t) > \theta_{n'}$, then, for every n', we have $f(u, t) = 0$. Hence $f(u, t) = 0$ if $\min(u, t) < \theta_{n'}$ and $\max(u, t) > \bar{v}$. By passage to the limit with n', we get $f(u, t) = 0$ if $\min(u, t) < \bar{v}$ and $\max(u, t) > \bar{v}$, i.e., $\bar{v} \in \Xi$.

Proof of Theorem 6. According to Lemma 4, the set $T \setminus \Xi$ is a set-theoretical union of disjoint open intervals (a_i, β_i). If $u_0 \in (a_{i_1}, \beta_{i_1})$ and $t_0 \in (a_{i_2}, \beta_{i_2})$, where $i_1 \neq i_2$, and $u < t$, then $f(u_0, t_0) = 0$, because there exists a $\bar{v}_\theta$ with $\beta_{i_1} \leqslant \bar{v}_\theta \leqslant a_{i_2}$ such that $f(u, t) = 0$ for $\min(u, t) < \bar{v}_\theta$ and $\max(u, t) > \bar{v}_\theta$. So the point (u_0, t_0) satisfies these conditions. Examining the set of those (u, t) for which $f(u, t) = 0$, it suffices to investigate them within the intervals (a_i, β_i) separately. So, in order to prove that the form of solution of equation (3) coincides with the form occurring in the assertion of our theorem, it suffices to apply the arguments of the proof of Theorem 4 to each of the intervals (a_i, β_i), but taking closed or half-closed intervals instead of them if a_i or β_i are such that $f(a_i, t) \not\equiv 0$ or $f(\beta_i, t) \not\equiv 0$.

Now we show that every function of this form is a solution of equation (3).

If $u_0 < v_0 < t_0$ and (u_0, v_0), (u_0, t_0) and (v_0, t_0) do not belong to A, the fulfillment of equation (3) is evident. If $(u_0, t_0) \in A$, then there exists a $\bar{v}$ with $u_0 \leqslant \bar{v} \leqslant t_0$ and $f(u, t) = 0$ for $\min(u, t) < \bar{v}$ and $\max(u, t) > \bar{v}$. If $v_0 \leqslant \bar{v}$, then $f(v_0, t_0) = 0$ and equation (3) is satisfied. The same goes for $v_0 > \bar{v}$. If $v_0 = \bar{v}$, then also $f(v_0, t_0) = 0$ or $f(u_0, t_0) = 0$. If $(v_0, t_0) \in A$, i.e., $f(v_0, t_0) = 0$, then $f(u_0, t_0) = 0$ and $f(u_0, v_0) = 0$ similarly implies $f(u_0, t_0) = 0$. In both cases equation (3) is satisfied which completes the proof of the theorem.

Next, consider the case where $f(u, u) \geqslant 0$ for $u \in T$. First, we prove

LEMMA 5. *The necessary and sufficient conditions for a Gaussian process* $\xi(t)$, $t \in T$, *to be Markovian are the following*:

(a) *a process* $\xi(t)$, *where* $t \in T \setminus \{t\colon f(t, t) = 0\}$, *is Markovian*;

(b) *for an arbitrary* $t_0 \in T$ *satisfying* $f(t_0, t_0) = 0$, *we have* $f(u, t) = 0$ *for* $u \leqslant t_0 \leqslant t$.

Proof. Necessity. The necessity of (a) is obvious. If $\xi_k \equiv O$, the

$$E(\xi_{k+1}|\xi_k, \xi_{k-1}, \ldots, \xi_1) = E(\xi_{k+1}|\xi_k) = 0 = P_{\xi_1, \xi_2, \ldots, \xi_{k-1}} \xi_{k+1}$$

which means that $E\xi_{k+1}\xi_i = 0$ for $i = 1, 2, \ldots, k-1$.

Sufficiency. If $\xi_k \not\equiv 0$, then

$$E(\xi_{k+1}|\xi_k, \ldots, \xi_1) = E(\xi_{k+1}|\xi_k, \xi_{i_1}, \ldots, \xi_{i_l}) = E(\xi_{k+1}|\xi_k),$$

where ξ_{i_j} are random variables which are not identically equal to zero.

If $\xi_k \equiv 0$, then condition (b) entails $E(\xi_{k+1}|\xi_k, \ldots, \xi_1) = 0$ and, simultaneously, $E(\xi_{k+1}|\xi_k) = 0$.

Write

$$A_1 = \bigcup_{\varphi \in \Phi} \{(u, t)\colon \min(u, t) \leqslant \varphi \text{ and } \max(u, t) \geqslant \varphi\},$$

where $\Phi = \{u\colon f(u, u) = 0\}$.

THEOREM 7. *Let* $\xi(t)$, $t \in T$, *be a Gaussian process and* $f(t, u)$ *its covariance function. Then* $\xi(t)$ *is a Markov process if and only if*

$$f(t, u) = \begin{cases} g(\min(u, t))\, h(\max(u, t)) & \text{for } (u, t) \notin A \cup A_1, \\ 0 & \text{for } (u, t) \in A \cup A_1, \end{cases} \tag{4}$$

where A *is the set defined in the assertion of Theorem* 5.

Proof. Arguing again as in the proof of Theorem 3, it is easy to see that the set $\Xi \cup \Phi$ is closed if $\xi(t)$ is Gaussian Markov process. Analogously as in the proof of Theorem 6, the required form of the covariance function $f(u, t)$ is obtained. If $f(t, u)$ is of form (4), then, for $t \in T$, the functional equation (3) is satisfied, i.e., the conditions of Lemma 5 are fulfilled, and thus such a process is Markovian.

Now we investigate what conditions should be imposed on the functions g and h in order that the function of form (4) be the covariance function of a Gaussian process.

THEOREM 8. *A function of form* (4) *is the covariance function of a Gaussian process if and only if the functions* g *and* h *satisfy the condition*

$$\frac{g(t)}{h(t)} \geqslant \frac{g(u)}{h(u)} \quad \text{for } t \geqslant u.$$

Proof. The function $f(t, u)$ is the covariance function of a Gaussian process if and only if, for an arbitrary sequence $t_1 < t_2 < \ldots < t_n$ of elements of T,

$$\det \Lambda(t_1, \ldots, t_n) = \det \begin{bmatrix} f(t_1, t_1) & f(t_1, t_2) & \ldots & f(t_1, t_n) \\ f(t_2, t_1) & f(t_2, t_2) & \ldots & f(t_2, t_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ f(t_{n-1}, t_1) & f(t_{n-1}, t_2) & \ldots & f(t_{n-1}, t_n) \\ f(t_n, t_1) & f(t_n, t_2) & \ldots & f(t_n, t_n) \end{bmatrix} \geqslant 0.$$

We restrict our attention to the case of $f(t, u) \neq 0$. Then, as one can easily evaluate

$$\det \Lambda(t_1, t_2, \ldots, t_n) = g_1(g_2 h_1 - g_1 h_2)(g_3 h_2 - g_2 h_3) \ldots (g_n h_{n-1} - g_{n-1} h_n) h_n,$$

where $g_i = g(t_i)$ and $h_i = h(t_i)$ for $i = 1, 2, \ldots, n$.

There should hold the following inequality:

$$g_1(g_2 h_1 - g_1 h_2)(g_3 h_2 - g_2 h_3) \ldots (g_n h_{n-1} - g_{n-1} h_n) h_n \geqslant 0.$$

Divide its both sides by $g_1 h_1 h_2^2 h_3^2 \ldots h_n^2 > 0$. We get

$$\left(\frac{g_2}{h_2} - \frac{g_1}{h_1}\right)\left(\frac{g_3}{h_3} - \frac{g_2}{h_2}\right) \ldots \left(\frac{g_n}{h_n} - \frac{g_{n-1}}{h_{n-1}}\right) \geqslant 0. \tag{5}$$

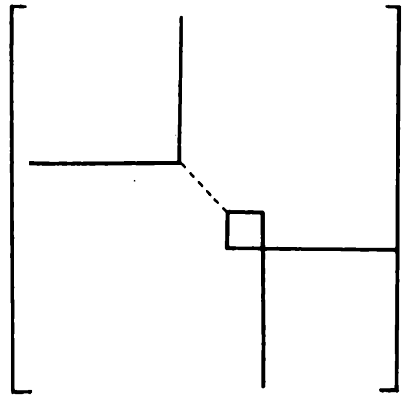
Simultaneously, we must have, for an arbitrary $k \leqslant n$,

$$g_{k-1}(g_k h_{k-1} - g_{k-1} h_k) h_k \geqslant 0 \text{ iff } \frac{g_k}{h_k} - \frac{g_{k-1}}{h_{k-1}} \geqslant 0.$$

The last condition is necessary and sufficient for inequality (5) to hold. It can be written down in the form

$$\frac{g(t)}{h(t)} \geqslant \frac{g(u)}{h(u)} > 0 \quad \text{for } t \geqslant u.$$

Clearly, if we admit $f(t, u) = 0$, the matrix $\Lambda(t_1, t_2, \ldots, t_n)$ takes the form



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where rectangles denote matrices as in the case $f(t, u) \neq 0$. Such matrices are positive-definite if and only if the matrices marked in the figure by rectangles are positive-definite, and this is equivalent to the condition which has been derived recently.

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Reçu par la Rédaction le 8. 12. 1972