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A NOTE ON COLLECTIONWISE NORMALITY AND PRODUCT SPACES

BY

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1. Introduction. A T_1-space is *λ-collectionwise normal* if for every discrete family $\{F_a\}_{a<\lambda}$ of closed subsets of X there exists a family $\{V_a\}_{a<\lambda}$ of disjoint open sets such that $F_a \subset V_a$. It is well known that normality is equivalent to $\aleph_0$-collectionwise normality. Bing [1] gave an example of a normal space which is not $\aleph_1$-collectionwise normal. Gantner [6], Starbird [11] and also Nedev raised the question whether there exists for every infinite cardinal λ a λ-collectionwise normal space which is not λ^+-collectionwise normal. Blair [2] proved that under the assumption of the Generalized Continuum Hypothesis there exists such a space for every infinite regular cardinal number. In this note we give – without any set-theoretic assumptions – the affirmative answer to that question.

THEOREM 1. *For every infinite cardinal λ there exists a (perfectly normal, metacompact) space which is λ-collectionwise normal but not λ^+-collectionwise normal.*

Let $w(K)$ denote the *weight* of the space K.

COROLLARY 1. *For every infinite cardinal λ there exists a (perfectly normal, metacompact) space X_λ such that, for every compact K,*

$$X_\lambda \times K \text{ is normal} \Leftrightarrow w(K) \leqslant \lambda.$$

Theorem 1 and Corollary 1 are consequences of more general results proved in Section 2.

In [5] Fleissner gave an example of a normal, collectionwise Hausdorff non-collectionwise normal space. (A T_1-space is *collectionwise Hausdorff* if for every discrete collection $\{p_s\}_{s\in S}$ of points there exists a disjoint collection $\{V_s\}_{s\in S}$ of open sets such that $p_s \in V_s$ for $s \in S$.) In Section 3 we present another description of essentially the same example. Our construction seems to be simpler and more natural.

2. λ-collectionwise normal spaces. Theorem 1 and Corollary 1 are immediate consequences of the following more general results. It suffices to put $\tau = \lambda^+$.

THEOREM 2. *For an infinite cardinal τ there exists a (perfectly normal, metacompact) non-τ-collectionwise normal space which is λ-collectionwise normal for every $\lambda < \tau$ if and only if τ is regular and uncountable.*

COROLLARY 2. *For an infinite cardinal τ there exists a space X such that, for every compact K,*

$$X \times K \text{ is normal} \Leftrightarrow w(K) < \tau$$

if and only if τ is regular.

Moreover, such a space may be assumed to be perfectly normal and metacompact iff τ is uncountable.

Proof of Theorem 2. It is easy to observe that if τ is countable or irregular and the space X is λ-collectionwise normal for every $\lambda < \tau$, then X is τ-collectionwise normal.

Let τ be an uncountable regular cardinal number. The construction of the space X with the desired properties is based on Engelking's exposition of Bing's example ([4], Example 5.1.23). For every cardinal number $\theta \leqslant \tau$ let D_θ be a discrete space of cardinality θ and denote by $\mathscr{K}$ the family of all coverings $\mathscr{F}$ of $D = D_\tau$ consisting of disjoint sets and of cardinality less than τ. For every $\mathscr{F} \in \mathscr{K}$ let $f_{\mathscr{F}}$ be a one-to-one function of $\mathscr{F}$ onto $D_{|\mathscr{F}|}$ and let $g_{\mathscr{F}}\colon D \to D_{|\mathscr{F}|}$ be defined by $g_{\mathscr{F}}(x) = f_{\mathscr{F}}(F)$ if $x \in F \in \mathscr{F}$. The diagonal mapping

$$G = \mathop{\Delta}_{\mathscr{F} \in \mathscr{K}} g_{\mathscr{F}}$$

is a homeomorphic embedding of D into

$$Y = \prod_{\mathscr{F} \in \mathscr{K}} D_{|\mathscr{F}|}$$

(cf. [4], The Diagonal Theorem 2.3.20), so that we can identify D with $G(D) \subset Y$. Let $X = Y_D$ (cf. [4], Example 5.1.22), i.e. X is the set Y with the topology obtained from the topology of Y by means of making the points of $Y \setminus D$ isolated.

I. *X is λ-collectionwise normal for every $\lambda < \tau$.*

Let $\{F_\alpha\}_{\alpha<\lambda}$ be a discrete family of closed sets in X. The covering

$$\mathscr{F}_0 = \{F_\alpha \cap D\}_{\alpha<\lambda} \cup \{D \setminus \bigcup_{\alpha<\lambda} F_\alpha\}$$

belongs to $\mathscr{K}$ and the sets

$$U_\alpha = \Pi_{\mathscr{F}_0}^{-1}\big(f_{\mathscr{F}_0}(F_\alpha \cap D)\big), \quad \text{where } \alpha < \lambda \text{ and } \Pi_{\mathscr{F}_0}\colon \prod_{\mathscr{F} \in \mathscr{K}} D_{|\mathscr{F}|} \to D_{|_0\mathscr{F}|}$$

is the projection, are open and disjoint in X and $F_a \cap D \subset U_a$. The sets

$$V_a = \Big(U_a \setminus \bigcup_{\substack{\beta<\lambda \\ \beta\neq a}} F_\beta\Big) \cup F_a$$

are also open and disjoint in X and $F_a \subset V_a$.

II. *X is not τ-collectionwise normal.*

The family of one-point subsets of D is discrete in X and has cardinality τ. To show that X is not τ-collectionwise normal it suffices to prove that every family of non-empty disjoint open sets in Y has cardinality less than τ.

Definition. A space Z has *calibre* τ if, for every family $\mathscr{U}$ of open subsets of Z of cardinality τ, there exists a subfamily $\mathscr{V}$ of cardinality τ with $\bigcap \mathscr{V} \neq \emptyset$.

It is clear that every collection of disjoint open sets in a space having calibre τ has cardinality less than τ. As every space D_λ for $\lambda < \tau$ has calibre τ, it follows from the theorem of Šanin ([12], see also [3], Theorem 11) that Y has calibre τ, which completes the proof of II.

Michael [7] modified Bing's example and obtained a perfectly normal metacompact space, which is normal but not $\aleph_1$-collectionwise normal. In an analogous way we can modify our space X and construct a perfectly normal metacompact non-τ-collectionwise normal space which is λ-collectionwise normal for every $\lambda < \tau$. Assume that $D_\lambda = \{a\colon a < \lambda\}$ for $\lambda < \tau$ and that for every point $d \in D$ and the covering $\mathscr{F}_d = \big\{\{d\}, D\setminus\{d\}\big\}$ we have $f_{\mathscr{F}_d}(\{d\}) = 0$. Let

$$\hat{X} = D \cup \{x = \{x_{\mathscr{F}}\}_{\mathscr{F}\in\mathscr{K}} \in X\colon x_{\mathscr{F}} \neq 0 \text{ for all } \mathscr{F} \in \mathscr{K} \text{ except for a finite number}\}.$$

One can easily check that the subspace $D \times \{0\} \cup (\hat{X}\setminus D) \times \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ of the product space $X \times \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ has the required properties.

A T_2-space X is *λ-paracompact* if every open cover of X of cardinality not greater than λ admits an open locally finite refinement. By I^λ we denote the Tychonoff cube of weight λ. In the sequel we shall make use of the following results:

THEOREM 3 (Morita [8]). *$X \times I^\lambda$ is normal if and only if X is normal and λ-paracompact.*

THEOREM 4 (Rudin [10]). *If K is compact and $X \times K$ is normal, then X is $w(K)$-collectionwise normal.*

Proof of Corollary 2. For $\tau = \aleph_0$ the existence of a space with the desired properties is equivalent to the existence of a normal space which is not countably paracompact. This is a consequence of Theorem 3 and of the well-known fact that if K is compact and non-discrete and $X \times K$

is normal, then X is countably paracompact (cf. [4], the proof of Theorem 5.2.8). Rudin [9] gave an example of such a space.

If τ is regular and uncountable, then it follows from Theorem 2 that there exists a metacompact non-τ-collectionwise normal space which is λ-collectionwise normal for every $\lambda < \tau$. Such a space is λ-paracompact for $\lambda < \tau$ (cf. [4], the proof of Theorem 5.3.3). As every compact space of weight λ is a closed subspace of I^λ, we infer from Theorems 3 and 4 that X has the required properties.

Finally, assume that τ is irregular and that for every compact K with $w(K) < \tau$ the space $X \times K$ is normal. By Theorem 3, X is normal and λ-paracompact for $\lambda < \tau$. Hence, as one can easily check, X is τ-paracompact and, consequently, $X \times I^\tau$ is normal.

To complete the proof, it suffices to recall that every perfectly normal or normal and metacompact space is countably paracompact ([4], Corollary 5.2.5, and Theorem 5.2.6).

Remark. It would be nice if we could assign to every normal space X a cardinal number τ such that for every compact K the product space $X \times K$ be normal if and only if $w(K) < \tau$. Unfortunately, generally no such τ exists. Indeed, let X be the space of ordinals less than ω_1, K_1 — the space of ordinals not greater than ω_1, and K_2 — the lexicographically ordered square. Then $w(K_1) = \aleph_1$, $w(K_2) = \mathfrak{c}$, and $X \times K_2$ is normal, though $X \times K_1$ is not (for similar considerations see [10]).

3. A normal collectionwise Hausdorff non-collectionwise normal space. Let $D = \{\beta\}_{\beta<\omega_1}$ be the discrete space of cardinality $\aleph_1$ and let ω_1 be the space of countable ordinals. Denote by $\mathcal{T}$ the family of all open-and-closed subsets of the subspace $Z = \{(\alpha, \beta) \in \omega_1 \times D: \beta < \alpha\}$ of the product space $\omega_1 \times D$. For every $T \in \mathcal{T}$ let f_T: $Z \to \{0, 1\}$ be the characteristic function of T and define the mapping f_0: $Z \to \omega_1$ by $f_0(\alpha, \beta) = \alpha$. The diagonal mapping

$$F = f_0 \Delta \mathop{\Delta}_{T \in \mathcal{T}} f_T$$

is a homeomorphic embedding of Z into $Y = \omega_1 \times \{0, 1\}^{\mathcal{T}}$ considered with the Tychonoff topology (cf. [4], The Diagonal Theorem 2.3.20), so that we can identify Z with $F(Z) \subset Y$. Let $X = Y_Z$ (cf. [4], Example 5.1.22), i.e. X is the set Y with the topology obtained from the topology of Y by means of making the points of $Y \setminus Z$ isolated.

I. *X is normal.*

Let A and B be disjoint and closed in X. As the points of $X \setminus Z$ are isolated, we may assume that A and B are contained in Z. There exists $T \in \mathcal{T}$ such that $A \subset T \subset Z \setminus B$. The set $U = \omega_1 \times \Pi_T^{-1}(1)$ is open-and-closed in X and $A \subset U \subset X \setminus B$. Here Π_T: $\{0, 1\}^{\mathcal{T}} \to \{0, 1\}_T$ denotes the natural projection.

Definition. A set of the form $\bigcap_{i=1}^{n} \Pi_{T_i}^{-1}(e_i)$, where $T_i \in \mathcal{T}$ and $e_i \in \{0, 1\}$, is said to be of *degree n*.

The following facts are well known (cf. [5]) and can be easily checked:

FACT 1. *If $g: \omega_1 \to \omega_1$, then the set $C = \{\alpha: \beta < \alpha$ implies $g(\beta) < \alpha\}$ is uncountable and closed in ω_1.*

FACT 2. *If $g: \omega_1 \to \omega_1$ and $g(\alpha) < \alpha$ for every $\alpha \neq 0$, then there is a cofinal subset L of ω_1 such that $L = g^{-1}(\beta)$ for some $\beta \in \omega_1$.*

FACT 3. *There are at most 2^n non-empty disjoint sets of degree n.*

II. *X is collectionwise Hausdorff.*

Let A be a discrete subset of X. We may obviously assume that $A \subset Z$. Denote by $\Pi_0: \omega_1 \times \{0, 1\}^{\mathcal{T}} \to \omega_1$ the projection and for every $\beta \in \omega_1$ put $g(\beta) = \max\{\alpha: (\alpha, \beta) \in A\}$ or $g(\beta) = 0$ if there is no such α. From Fact 1 we infer that the set $C = \{\alpha: \beta < \alpha$ implies $g(\beta) < \alpha\}$ is closed and uncountable in ω_1. Then

$$\omega_1 \setminus C = \bigcup_{\lambda < \omega_1} V_\lambda,$$

where V_λ are order components of $\omega_1 \setminus C$, and hence are open in ω_1, disjoint and countable. Note that $\Pi_0^{-1}(\alpha) \cap A = \emptyset$ for every $\alpha \in C$. The sets $\Pi_0^{-1}(V_\lambda)$, $\lambda < \omega_1$, are open and disjoint in X,

$$A \subset \bigcup_{\lambda < \omega_1} \Pi_0^{-1}(V_\lambda) \quad \text{and} \quad |\Pi_0^{-1}(V_\lambda) \cap A| \leqslant \aleph_0.$$

It follows that the points of A can be separated by open sets in X.

III. *X is not collectionwise normal.*

Let $Z_\beta = \{(\alpha, \beta): \beta < \alpha < \omega_1\}$. Then the family $\{Z_\beta\}_{\beta \in \omega_1}$ is discrete in X. Assume that there exist open and disjoint in X sets G_β such that $Z_\beta \subset G_\beta$. For every β and $\alpha > \beta$ there exist an ordinal number $\varrho(\alpha, \beta)$ and a non-empty set $U(\alpha, \beta)$ of degree $n(\alpha, \beta)$ in $\{0, 1\}^{\mathcal{T}}$ such that

$$\beta \leqslant \varrho(\alpha, \beta) < \alpha \quad \text{and} \quad \big(\varrho(\alpha, \beta), \alpha\big] \times U(\alpha, \beta) \subset G_\beta.$$

From Fact 2 we infer that for every β there exist a $\varrho_\beta \geqslant \beta$, $n_\beta \in \omega_0$ and a cofinal subset $L_\beta \subset \omega_1$ such that for every $\alpha \in L_\beta$ we have $\varrho(\alpha, \beta) = \varrho_\beta$ and $n(\alpha, \beta) = n_\beta$. There exist an $n \in \omega_0$ and 2^n+1 ordinal numbers $\beta_i \in \omega_1$ such that $n_{\beta_i} = n$ for $i = 1, 2, \ldots, 2^n+1$. Let

$$\varrho = \max\{\varrho_{\beta_i}+1: i = 1, 2, \ldots, 2^n+1\}.$$

We conclude that for every $i \leqslant 2^n+1$ there is a non-empty set U_{β_i} of degree n such that $\{\varrho\} \times U_{\beta_i} \subset G_{\beta_i}$. Hence the sets $\{U_{\beta_i}\}_{i=1}^{2^n+1}$ are disjoint, which is impossible by Fact 3.

Remark. Though the space X has character $2^{\aleph_1}$, it contains a subspace X^* of character $2^{\aleph_0}$ with the same properties. Indeed, write A_α

$= \{(\gamma, \delta) \in Z \colon \gamma \leqslant a\}$ and let $\underset{a}{\sim}$ be the equivalence relation in $\mathscr{T}$ defined by letting $T_1 \underset{a}{\sim} T_2$ if and only if $T_1 \cap A_a = T_2 \cap A_a$. It is easy to check, in an analogous way, that the subspace

$$X^* = \{(a, \{x_T\}_{T \in \mathscr{T}}) \in X \colon \text{for every } T_1, T_2 \in \mathscr{T}, \text{ if } T_1 \underset{a}{\sim} T_2, \text{ then } x_{T_1} = x_{T_2}\}$$

contains Z, is normal and collectionwise Hausdorff, but not collectionwise normal. Its character is equal to $2^{\aleph_0}$, as the relation $\underset{a}{\sim}$ has at most $2^{\aleph_0}$ different equivalence classes.

Added in proof. For another description of the space considered in Theorem 1 and for some applications of that theorem see the author's papers *Collectionwise normality and absolute retracts* and *Collectionwise normality and extensions of continuous functions and pseudometrics* to appear in Fundamenta Mathematicae.

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