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TWO COUNTER-EXAMPLES CONCERNING HAUSDORFF DIMENSIONS OF PROJECTIONS

BY

ROY O. DAVIES (LEICESTER)

The problem. Let E be an analytic plane set of (Hausdorff) dimension s. The following two theorems were proved by Marstrand [6].

I. *If $s > 1$, then E projects into a (linear) set of positive Lebesgue measure in almost all directions.*

I*. *If $s \leqslant 1$, then E projects into a set of dimension s in almost all directions.*

It would be agreeable if such results could be extended to arbitrary plane sets. We show in this paper, by means of two counter-examples constructed using the continuum hypothesis, that they cannot be extended even to arbitrary Λ^s-measurable sets. The proofs owe a great deal to discussions with J. M. Marstrand.

Marstrand's theorems have recently been generalized to higher dimensions by Mattila [9], and no doubt the same could be done for our counter-examples.

Essentially s-dimensional sets. A continuous increasing function $\varphi\colon [0, \infty) \to [0, \infty)$ is a *measure function* if $\varphi(0) = 0$ and $\varphi(x) > 0$ when $x > 0$. We define the Hausdorff measure $\Lambda^\varphi(E)$ for any plane set E as $\lim_{\delta \to 0+} \Lambda_\delta^\varphi(E)$, where $\Lambda_\delta^\varphi(E)$ is the infimum of $\sum_n \varphi(dE_n)$ for all possible coverings of E by a sequence (E_n) of sets of diameters $dE_n \leqslant \delta$. When $\varphi(x) = x^s$, $s > 0$, one writes Λ^s for Λ^φ, and simply Λ when $s = 1$. The *(Hausdorff) dimension* of E is the unique number s_0 such that $\Lambda^s(E) = 0$ for every $s > s_0$ and $\Lambda^s(E) = \infty$ for every $s < s_0$. We shall call a set *essentially at least s-dimensional* if it cannot be expressed as the union of countably many sets each of dimension less than s. For obvious reasons we omit the words "at least" for plane sets when $s = 2$, and for linear sets when $s = 1$. We need the following two lemmas, the first of which is trivial.

LEMMA 1. *If there exists a measure function φ such that $\Lambda^{\varphi}(E) > 0$ and $\varphi(x) = o(x^t)$ as $x \to 0$ for each $t < s$, then E is essentially at least s-dimensional.*

LEMMA 2. *The continuum hypothesis implies that every essentially at least s-dimensional set E has an essentially at least s-dimensional subset E_0 which is Λ^t-measurable for every value of $t < s$.*

Proof. We apply the standard method used by Besicovitch [1] in constructing a "rarefied" set. List all G_δ-sets of dimension less than s as K_a, $0 \leqslant a < \omega_1$. For $0 \leqslant a < \omega_1$ define q_a as any point of E not in the set $\bigcup\{K_\beta\colon 0 \leqslant \beta \leqslant a\}$; such a point exists, since the union is countable and, therefore, represents a set not essentially at least s-dimensional. We assert that the set $E_0 = \{q_a\colon 0 \leqslant a < \omega_1\}$ has the desired properties.

First, if $H^1, H^2, \ldots$ is any sequence of sets each of dimension less than s, we can include H^i in a G_δ-set K^i of the same dimension, and K^i is listed as, say, $K_{a(i)}$, where $0 \leqslant a(i) < \omega_1$. The ordinals $a(i)$ have an upper bound $a < \omega_1$ and the point q_a is in the set $E_0 - \bigcup_i H^i$ which is, therefore, non-empty. Hence E_0 is essentially at least s-dimensional.

Second, as pointed out by Besicovitch [1], for a set E_0 to be Λ^t-measurable it is necessary and sufficient that, for every G_δ-set K of finite Λ^t-measure,

$$\Lambda^t(K) = \Lambda^t(K \cap E_0) + \Lambda^t(K - E_0). \tag{1}$$

But if $t < s$, then any such set K is listed as, say, K_a, and $K \cap E_0$ is a part of the countable set $\{q_\beta\colon 0 \leqslant \beta < a\}$. Hence (1) holds, and thus E_0 is Λ^t-measurable.

LEMMA 2*. *If, in addition, E is Λ^s-non-σ-finite, we may arrange that E_0 is also Λ^s-measurable.*

Proof. Additionally incorporate in the list of K_a's all G_δ-sets of finite Λ^s-measure, and then proceed as before.

Dimension and linear sections. Let E be a plane set.

LEMMA 3. *If $l \cap E$ is essentially one-dimensional for all vertical lines l through the points of some set J on the x-axis of positive linear measure, then E is essentially two-dimensional.*

Proof. Let (X_n) be any sequence of sets each of dimension less than 2, and let Y_n denote the set of points x on the x-axis such that the vertical line $l(x)$ through x meets X_n in a set of dimension one. By Marstrand's theorem [7], the set Y_n is of measure zero. Hence there exists a point $x \in J - \bigcup_n Y_n$, and now $l(x) \cap \left(\bigcup_n X_n\right)$ is not essentially one-dimensional. Hence $\bigcup_n X_n$ cannot contain E.

COROLLARY. *The result still holds if we replace vertical lines through the points of a set of positive measure on the x-axis by lines touching the circle $x^2+y^2=1$ at a set of positive measure. The same applies to Lemma* 3* *below.*

Proof. This follows from the fact that, in a sufficiently small neighbourhood of any point outside the circle, the new situation can sufficiently smoothly be deformed into the old one.

LEMMA 3*. *If $l\cap E$ is uncountable for all vertical lines l through the points of some set J on the x-axis of positive linear measure, then E is Λ-non-σ-finite.*

Proof. This can be deduced in an obvious way from the fact that if $\Lambda(X)$ is finite, then $l(x)\cap X$ is countable for almost all $x\in \boldsymbol{R}^1$. In proving the latter we may suppose that X is compact, since

$$X\subseteq \Big(\bigcup_n K_n\Big)\cup Z,$$

where each K_n is compact with $\Lambda(K_n)<\infty$ and $\Lambda(Z)=0$. The result now follows from the observation that if $A\subseteq \boldsymbol{R}^1$ and $K\subseteq \boldsymbol{R}^2$ are compact and for each $x\in A$ the set $l(x)\cap K$ contains k points in, for example, the respective strips $2i-1\leqslant y\leqslant 2i$ $(i=1,\dots,k)$, then K contains k disjoint compact sets the projection of each of which on the x-axis contains A, and hence $\Lambda(K)\geqslant k\Lambda(A)$.

Two anomalous linear sets. The next two results are related to the problems of Marczewski [4], [5] discussed in [2], which contains Lemma 4* for linear transformations; the proof of Lemma 4 is similar to an argument in [8] and was suggested by Marstrand. (He pointed out the inadequacy of my original version of Lemma 4, proved, using Theorem 6 of Eggleston [3], only for linear transformations.) A non-degenerate bilinear transformation is of the form

$$t\colon x \rightsquigarrow (ax+b)/(cx+d),\qquad \text{where } ad-bc\neq 0,$$

and may be regarded as either from the projective real line onto itself or from $\boldsymbol{R}-\{-d/c\}$ into $\boldsymbol{R}$; in the latter case $t(E)$ means $t(E-\{d/c\})$.

LEMMA 4. *There exists a linear set M of linear measure zero such that for every sequence (t_n) of non-degenerate bilinear transformations the set $\bigcap_{n=1}^{\infty} t_n(M)$ is essentially one-dimensional.*

Proof. As is known, there exists a compact linear set A of linear measure zero but essentially one-dimensional. We may and shall identify the points of the projective real line (regarded as the x-axis together with a point at infinity) with their images on the circle $C\colon x^2+(y-1)^2=1$ under projection from the point $(0,2)$; this preserves compactness, measure zero, and essential one-dimensionality. The non-degenerate bilinear transformations are now regarded as acting on C, and form a locally compact separable topological transformation group G such that if $g \in G$, then, given $\varepsilon > 0$, there is a neighbourhood N of g with $\Lambda(NA) < \varepsilon$.

Let $\{g_1, g_2, \ldots\}$ be a countable dense subset of G, and for each i choose a neighbourhood N_i of g_i such that $\Lambda(N_i A) < 2^{-i}$. We show that the set

$$M = \limsup_{i\to\infty} N_i A$$

has the asserted properties.

Evidently, $\Lambda(M) = 0$. For any sequence (t_n) of elements of G, the set

$$\bigcap_{n=1}^{\infty}\bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty} t_n N_j$$

is a countable intersection of dense open subsets of G, and hence is non-empty; let x be an element of it. Then

$$xA \subseteq \bigcap_{n=1}^{\infty} t_n\Big(\bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty} N_j A\Big) = \bigcap_{n=1}^{\infty} t_n M$$

and, consequently, the set on the right-hand side is essentially one-dimensional.

LEMMA 4*. *There exists a linear set M^* of dimension zero such that for any sequence (t_n) of non-degenerate bilinear transformations the set $\bigcap_{n=1}^{\infty} t_n(M^*)$ is uncountable.*

Proof. We may take

$$M^* = \limsup_{n\to\infty} F_n,$$

where F_n consists of closed intervals of length n^{-3n} equally spaced out along the whole of R^1 in such a way that there are n of them in $I = [0, 1]$, including $[0, n^{-3n}]$. One easily verifies that $\Lambda_{1/n}^{1/n}[F_n \cap I] \leqslant n^{-2}$ and that M^* is co-meagre (residual).

Two anomalous plane sets. We write P^2 for R^2 together with a point at infinity in every direction. If $p \in P^2$ and $E \subseteq R^2$, then $p(E)$ denotes the union of all lines (in R^2) joining p to points of $E - \{p\}$; or, if p is at infinity, then $p(E)$ denotes the union of all lines through points of E in the direction of p. In the sequel we shall not have to distinguish points at infinity and ordinary points. We say that *the projection of E from p is of linear measure zero* if this is true of the intersection of $p(E)$ with some (and therefore any) line not through p; similarly we may speak of the projection being of dimension zero.

THEOREM 1. *Under the continuum hypothesis, there exists an essentially two-dimensional plane set E whose projection from every point of P^2 is of linear measure zero.*

Proof. We may list all points of P^2 as p_a, $0 \leqslant a < \omega_1$, and all lines touching the circle $x^2+y^2 = 1$ as l_a, $0 \leqslant a < \omega_1$, in such a way that l_a avoids both p_a and (if $a > 0$) some point p_β with $\beta < a$. Let M_0 be a congruent copy on l_0 of the set M of Lemma 4, and define M_a for $0 < a < \omega_1$ by transfinite induction by taking $M_a = l_a \cap \bigcap p_\beta(M_\beta)$, where β runs over all ordinals less than a for which $p_\beta \notin l_a$. We show that the set

$$E = \bigcup_a M_a$$

has the asserted properties.

In order to prove that E is essentially two-dimensional, we first note that each set M_a is essentially one-dimensional. Indeed, it is easy to see using transfinite induction that it is congruent to a set of the form $\bigcap_n t_n(M)$, where (t_n) is a sequence of non-degenerate bilinear transformations depending on a. We then apply the Corollary to Lemma 3.

Any point of P^2 is listed as, say, p_γ. Now for each a the set M_a is of (linear) measure zero (being part of a copy of M). Hence the set $l_\gamma \cap p_\gamma\big(\bigcup_{a<\gamma} M_a\big)$ is the union of countably many sets of measure zero, and so is itself of measure zero. If $a \geqslant \gamma$ and $p_\gamma \notin l_a$, we have $M_a \subseteq p_\gamma(M_\gamma)$ by the definition of M_a and, therefore, $l_\gamma \cap p_\gamma(M_a) \subseteq M_\gamma$; moreover, p_γ is on l_a for at most two values of a. Hence $l_\gamma \cap p_\gamma(E)$ is contained in the set

$$l_\gamma \cap p_\gamma\big(\bigcup_{a<\gamma} M_a\big) \cup M_\gamma,$$

together with at most two additional points, and is of measure zero. Therefore, the projection of E from p_γ is of measure zero. This completes the proof.

Remark. By applying Lemma 2 we can obtain a subset of E which has the same properties and which is, in addition, Λ^t-measurable for every positive value of t.

THEOREM 1*. *Under the continuum hypothesis, there exists a Λ-non-σ-finite plane set E^* whose projection from every point of P^2 is of dimension zero; and E^* may be supposed Λ^t-measurable for every positive value of t.*

The proof is as for Theorem 1, but based on the set M^* of Lemma 4* instead of on M, and applying the starred lemmas.

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