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ON CLASSES OF ALGEBRAS DEFINABLE BY REGULAR EQUATIONS

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1. In [3] J. Płonka investigated the smallest class of algebras which contains a given equational class and is defined by regular equations. Especially he considered the connection between such a class and the formation of sums of direct systems of algebras defined in [2] under the assumption that all algebras have no nullary fundamental operations.

In this note we shall give a complete description of such a closure defined by regular equations for arbitrary classes of algebras and without any assumption on the fundamental operations (Theorem 1). This will lead to a characterization of the corresponding closure operator using the notion of the sum of a direct system in the case that no nullary fundamental operations occur (Theorem 2).

2. All algebras under consideration are of finitary type $\Delta = (n_t)_{t\in T}$ and we write $\mathfrak{A}(\Delta)$ for the class of all algebras of type Δ, where an algebra of type Δ is a pair $\boldsymbol{A} = \big(A, (f_t)_{t\in T}\big)$, f_t being an n_t-ary operation on the carrier set A for each $t\in T$.

An equation $p = q$ (p and q can be considered as elements of the free algebra in $\mathfrak{A}(\Delta)$ generated by a set $\{x_i \mid i\in \boldsymbol{N}\}$ of "variables") is called *regular* if p contains the same variables as q.

If $\mathfrak{A} \subseteq \mathfrak{A}(\Delta)$, we write $\mathrm{Eq}(\mathfrak{A})$ for the set of all equations and $\mathrm{Eq}^{\mathrm{reg}}(\mathfrak{A})$ for the set of all regular equations which are valid in each algebra in $\mathfrak{A}$. For a set G of equations $\mathrm{Md}(G)$ is the class consisting of all algebras in $\mathfrak{A}(\Delta)$ in which every equation of G is valid.

It is clear that $\mathrm{MdEq}(\mathfrak{A})$ is the smallest class in $\mathfrak{A}(\Delta)$ which contains $\mathfrak{A}$ and is definable by equations and that $\mathrm{MdEq}^{\mathrm{reg}}(\mathfrak{A})$ is the smallest class in $\mathfrak{A}(\Delta)$ which contains $\mathfrak{A}$ and is definable by regular equations (MdEq and $\mathrm{MdEq}^{\mathrm{reg}}$ are closure operators on $\mathfrak{A}(\Delta)$).

By a well known theorem of G. Birkhoff, $\mathrm{MdEq}(\mathfrak{A})$ is equal to $\mathscr{H}\mathscr{S}\mathscr{P}(\mathfrak{A})$, where $\mathscr{H}$, $\mathscr{S}$, and $\mathscr{P}$ are the operators defined by forming all homomorphic images, all subalgebras, and all products of a given class of algebras [1]. The purpose of this paper is to state an analogous result for $\mathrm{MdEq}^{\mathrm{reg}}$.

3. For $\boldsymbol{A} = (A, (f_t)_{t\in T})$ we define the one-point extension $\dot{\boldsymbol{A}} = (\dot{A}, (\dot{f}_t)_{t\in T})$ of $\boldsymbol{A}$ by

$$\dot{A} := A \,\dot{\cup}\, \{0\} \quad \text{(disjoint union)},$$
$$\dot{f}_t(a_1, \ldots, a_{n_t}) = \begin{cases} f_t(a_1, \ldots, a_{n_t}) & \text{if } a_1, \ldots, a_{n_t} \in A, \\ 0 & \text{otherwise.} \end{cases}$$

For $\mathfrak{A} \subseteq \mathfrak{A}(\Delta)$ let $\dot{\mathscr{E}}(\mathfrak{A})$ be the class $\{\dot{\boldsymbol{A}} \mid \boldsymbol{A} \in \mathfrak{A}\}$.

Now we can prove the following

LEMMA 1. *Let* $\boldsymbol{A}$ *be a non-empty algebra in* $\mathfrak{A}(\Delta)$. *Then* $\mathrm{Eq}^{\mathrm{reg}}(\boldsymbol{A}) = \mathrm{Eq}(\dot{\boldsymbol{A}})$.

Proof. Let $p(x_1, \ldots, x_m) = q(x_1, \ldots, x_m) \in \mathrm{Eq}^{\mathrm{reg}}(\boldsymbol{A})$. Then for all $a_1, \ldots, a_m \in A$ we have $p(a_1, \ldots, a_m) = q(a_1, \ldots, a_m)$. For $a_1, \ldots, a_m \in \dot{A}$, where at least one a_i is equal to 0, we get $p(a_1, \ldots, a_m) = 0 = q(a_1, \ldots, a_m)$. Therefore $p = q \in \mathrm{Eq}(\dot{\boldsymbol{A}})$. Now let $p(x_1, \ldots, x_n) = q(y_1, \ldots, y_m)$ be an equation which is not regular. So assume without loss of generality $x_1 \notin \{y_1, \ldots, y_m\}$. Choosing an arbitrary $a \in A$ we get $p(0, a, \ldots, a) = 0$, $q(a, \ldots, a) \in A$ and, therefore, it follows $p = q \notin \mathrm{Eq}(\dot{\boldsymbol{A}})$. Since $\mathrm{Eq}(\dot{\boldsymbol{A}}) \subseteq \mathrm{Eq}(\boldsymbol{A})$, the proof is completed.

COROLLARY 1. *If* $\mathfrak{A} \subseteq \mathfrak{A}(\Delta)$ *and* $\boldsymbol{1}$ *is the one-element algebra in* $\mathfrak{A}(\Delta)$, *then* $\mathrm{Eq}^{\mathrm{reg}}(\mathfrak{A}) = \mathrm{Eq}(\dot{\mathscr{E}}(\mathfrak{A} \cup \{\boldsymbol{1}\}))$.

Proof. We have

$$\mathrm{Eq}^{\mathrm{reg}}(\mathfrak{A}) = \bigcap_{\boldsymbol{A}\in\mathfrak{A}} \mathrm{Eq}^{\mathrm{reg}}(\boldsymbol{A}) = \bigcap_{\substack{\boldsymbol{A}\in\mathfrak{A}\\ A\neq\emptyset}} \mathrm{Eq}^{\mathrm{reg}}(\boldsymbol{A}) \cap \mathrm{Eq}^{\mathrm{reg}}(\boldsymbol{1})$$
$$= \bigcap_{\substack{\boldsymbol{A}\in\mathfrak{A}\\ A\neq\emptyset}} \mathrm{Eq}(\dot{\boldsymbol{A}}) \cap \mathrm{Eq}(\dot{\boldsymbol{1}}) = \bigcap_{\boldsymbol{A}\in\mathfrak{A}} \mathrm{Eq}(\dot{\boldsymbol{A}}) \cap \mathrm{Eq}(\dot{\boldsymbol{1}}) = \mathrm{Eq}(\dot{\mathscr{E}}(\mathfrak{A} \cup \{\boldsymbol{1}\})).$$

As a simple consequence of this corollary we get

THEOREM 1. *Let* $\mathfrak{A} \subseteq \mathfrak{A}(\Delta)$. *Then* $\mathrm{MdEq}^{\mathrm{reg}}(\mathfrak{A}) = \mathscr{H}\mathscr{S}\mathscr{P}\dot{\mathscr{E}}(\mathfrak{A} \cup \{\boldsymbol{1}\})$.

Proof. $\mathrm{MdEq}^{\mathrm{reg}}(\mathfrak{A}) = \mathrm{MdEq}(\dot{\mathscr{E}}(\mathfrak{A} \cup \{\boldsymbol{1}\})) = \mathscr{H}\mathscr{S}\mathscr{P}\dot{\mathscr{E}}(\mathfrak{A} \cup \{\boldsymbol{1}\})$.

From this theorem there follows a simple criterion to decide whether an equational class can be defined by regular equations:

COROLLARY 2 [2]. *For an equational class* $\mathfrak{A} \subseteq \mathfrak{A}(\Delta)$ *the following conditions are equivalent*:

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[1] For this result and other details not proved in this note see [1].

[2] It has been communicated to me that this result is due to B. Jónsson and E. Nelson.

(i) $\mathfrak{A}$ *can be defined by regular equations.*

(ii) $\dot{\mathscr{E}}(\mathfrak{A}) \subseteq \mathfrak{A}$.

Proof. (i) is equivalent to $\mathrm{MdEq}^{\mathrm{reg}}(\mathfrak{A}) \subseteq \mathfrak{A}$, and because of $\boldsymbol{1} \in \mathfrak{A}$ we have $\mathrm{MdEq}^{\mathrm{reg}}(\mathfrak{A}) = \mathscr{H}\mathscr{S}\mathscr{P}\dot{\mathscr{E}}(\mathfrak{A})$. But from $\mathscr{H}\mathscr{S}\mathscr{P}\dot{\mathscr{E}}(\mathfrak{A}) \subseteq \mathfrak{A}$ it follows $\dot{\mathscr{E}}(\mathfrak{A}) \subseteq \mathfrak{A}$, and from $\dot{\mathscr{E}}(\mathfrak{A}) \subseteq \mathfrak{A}$ it follows $\mathscr{H}\mathscr{S}\mathscr{P}\dot{\mathscr{E}}(\mathfrak{A}) \subseteq \mathscr{H}\mathscr{S}\mathscr{P}(\mathfrak{A}) = \mathfrak{A}$.

4. In the sequel all algebras are of type $\Delta = (n_t)_{t\in T}$ with $n_t \neq 0$ for all $t \in T$.

First we repeat the definition of a direct system of algebras and its sum (see [2]):

Let $(I, \leqslant)$ be a partially ordered set with the property that for any two elements $i, j \in I$ there is the least upper bound l.u.b.(i, j) of i and j.

A *direct system* of algebras from $\mathfrak{A}(\Delta)$ over the set $(I, \leqslant)$ is a pair $\big((\boldsymbol{A}_i)_{i\in I}, (\varphi_{ij})_{i\leqslant j}\big)$, where $\boldsymbol{A}_i \in \mathfrak{A}(\Delta)$ for all $i \in I$ and $\varphi_{ij}\colon \boldsymbol{A}_i \to \boldsymbol{A}_j$ are homomorphisms for all $i, j \in I$ with $i \leqslant j$ such that

(a) $\varphi_{ii} = \mathrm{id}_{\boldsymbol{A}_i}$ for all $i \in I$,

(b) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ for all $i, j, k \in I$ with $i \leqslant j \leqslant k$.

The *sum* of the direct system is an algebra $\boldsymbol{S} = \big(S, (s_t)_{t\in T}\big)$, where $S = \dot{\bigcup\limits_{i\in I}} A_i$ (disjoint union) and $s_t(a_1, \ldots, a_{n_t}) = f_t\big(\varphi_{i_1 i_0}(a_1), \ldots, \varphi_{i_{n_t} i_0}(a_{n_t})\big)$, where $a_j \in A_{i_j}$ for $1 \leqslant j \leqslant n_t$ and $i_0 = \text{l.u.b.}(i_1, \ldots, i_{n_t})$.

We write $\mathscr{S}\!\mathscr{u}(\mathfrak{A})$ for the class of all sums of direct systems of algebras from $\mathfrak{A}$.

LEMMA 2. *Let* $\mathfrak{A} \subseteq \mathfrak{A}(\Delta)$. *Then*

(i) $\mathscr{P}\dot{\mathscr{E}}(\mathfrak{A}) \subseteq \mathscr{S}\!\mathscr{u}\mathscr{P}(\mathfrak{A})$,

(ii) $\mathscr{S}\!\mathscr{u}(\mathfrak{A}) \subseteq \mathscr{S}\mathscr{P}\dot{\mathscr{E}}(\mathfrak{A})$.

Proof. (i) Let $(\boldsymbol{A}_i)_{i\in I}$ be a family of algebras from $\mathfrak{A}$. We consider $\prod\limits_{i\in I} \dot{\boldsymbol{A}}_i$ and for $J \subseteq I$ the sets $B_J = \{(a_i)_{i\in I} \mid a_i = O_i \text{ for } i \in J,\ a_i \in A_i \text{ for } i \notin J\}$. It is easy to see that

$$\bigtimes_{i\in I} \dot{A}_i = \dot{\bigcup_{J\subseteq I}} B_J$$

and that B_J defines a subalgebra $\boldsymbol{B}_J$ of $\prod\limits_{i\in I} \dot{\boldsymbol{A}}_i$ which is isomorphic to $\prod\limits_{i\in I\setminus J} \boldsymbol{A}_i$. Now realize that $(\mathfrak{P}I, \subseteq)$ is a partially ordered set with the l.u.b.-property and that $\big((\prod\limits_{i\in I\setminus J} \boldsymbol{A}_i)_{J\in\mathfrak{P}I}, (\pi_{JJ'})_{J\subseteq J'}\big)$ is a direct system over $\mathfrak{P}I$, where $\pi_{JJ'}$ is the natural projection from $\prod\limits_{i\in I\setminus J} \boldsymbol{A}_i$ to $\prod\limits_{i\in I\setminus J'} \boldsymbol{A}_i$ whenever $J \subseteq J'$. Then it is easy to check that the sum of this direct system is isomorphic to $\prod_{i\in I}^{\bullet} \boldsymbol{A}_i$, where the isomorphism is induced by the bijection from $\dot{\bigcup}_{J\in I} B_J$ to $\dot{\mathsf{X}}_{i\in I} A_i$.

(ii) Let $\big((\boldsymbol{A}_i)_{i\in I},\ (\psi_{ij})_{i\leqslant j}\big)$ be a direct system of algebras from $\mathfrak{A}$ and let $\boldsymbol{S}$ be its sum. Now consider for each $k\in I$ the subalgebra $\boldsymbol{B}_k$ of $\prod_{i\in I}^{\bullet} \boldsymbol{A}_i$, where $B_k = \{(a_i)_{i\in I} \mid a_i = \psi_{ki}(a_k)$ for $k\leqslant i$ and $a_k\in A_k$, $a_i = O_i$ otherwise$\}$. Then $\boldsymbol{B}_k \simeq \boldsymbol{A}_k$ and these isomorphisms induce an obvious isomorphism between the subalgebra $\boldsymbol{U}$ of $\prod_{i\in I}^{\bullet} \boldsymbol{A}_i$ with $U = \bigcup_{k\in I} B_k$ and $\boldsymbol{S}$. From this it follows $\boldsymbol{S}\in \mathscr{S}\mathscr{P}\dot{\mathscr{E}}(\mathfrak{A})$.

Corollary 3. *$\mathscr{S}\mathscr{S}_u\mathscr{P}(\mathfrak{A})$ is a quasi-primitive class of algebras for any $\mathfrak{A} \subseteq \mathfrak{A}(\Delta)$.*

Proof. We have to show that $\mathscr{S}\mathscr{S}_u\mathscr{P}(\mathfrak{A})$ is closed with respect to the operators $\mathscr{S}$ and $\mathscr{P}$. First, $\mathscr{S}\mathscr{S}\mathscr{S}_u\mathscr{P}(\mathfrak{A}) \subseteq \mathscr{S}\mathscr{S}_u\mathscr{P}(\mathfrak{A})$ is a trivial statement. Now we consider $\mathscr{P}\mathscr{S}\mathscr{S}_u\mathscr{P}(\mathfrak{A})$. This class is contained in $\mathscr{S}\mathscr{P}\mathscr{S}_u\mathscr{P}(\mathfrak{A})$ and from Lemma 2, (ii), it follows $\mathscr{S}\mathscr{P}\mathscr{S}_u\mathscr{P}(\mathfrak{A}) \subseteq \mathscr{S}\mathscr{P}\mathscr{S}\mathscr{P}\dot{\mathscr{E}}\mathscr{P}(\mathfrak{A}) \subseteq \mathscr{S}\mathscr{P}\dot{\mathscr{E}}\mathscr{P}(\mathfrak{A})$. By (i) of Lemma 2 we get $\mathscr{S}\mathscr{P}\dot{\mathscr{E}}\mathscr{P}(\mathfrak{A}) \subseteq \mathscr{S}\mathscr{S}_u\mathscr{P}\mathscr{P}(\mathfrak{A}) \subseteq \mathscr{S}\mathscr{S}_u\mathscr{P}(\mathfrak{A})$, which completes the proof.

Finally, we characterize the operator $\mathrm{MdEq}^{\mathrm{reg}}$ for classes of algebras without nullary fundamental operations by

Theorem 2. *Let $\mathfrak{A} \subseteq \mathfrak{A}(\Delta)$. Then*

$$\mathrm{MdEq}^{\mathrm{reg}}(\mathfrak{A}) = \mathscr{H}\mathscr{S}\mathscr{S}_u\mathscr{P}(\mathfrak{A}).$$

Proof. By Lemma 2, (ii), we have $\mathscr{S}_u\mathscr{P}(\mathfrak{A}) \subseteq \mathscr{S}\mathscr{P}\dot{\mathscr{E}}\mathscr{P}(\mathfrak{A})$ and, therefore, $\mathscr{H}\mathscr{S}\mathscr{S}_u\mathscr{P}(\mathfrak{A}) \subseteq \mathscr{H}\mathscr{S}\mathscr{P}\dot{\mathscr{E}}\mathscr{P}(\mathfrak{A})$. Since a class defined by regular equations is closed with respect to the operators $\mathscr{H}$, $\mathscr{S}$, $\mathscr{P}$ (it is an equational class) and the operator $\dot{\mathscr{E}}$ (Lemma 1), $\mathscr{H}\mathscr{S}\mathscr{P}\dot{\mathscr{E}}\mathscr{P}(\mathfrak{A}) \subseteq \mathrm{MdEq}^{\mathrm{reg}}(\mathfrak{A})$, which proves the inclusion $\mathscr{H}\mathscr{S}\mathscr{S}_u\mathscr{P}(\mathfrak{A}) \subseteq \mathrm{MdEq}^{\mathrm{reg}}(\mathfrak{A})$.

From Lemma 2, (i), it follows that $\mathscr{P}\dot{\mathscr{E}}(\mathfrak{A}\cup\{\boldsymbol{1}\}) \subseteq \mathscr{S}_u\mathscr{P}(\mathfrak{A}\cup\{\boldsymbol{1}\}) = \mathscr{S}_u\mathscr{P}(\mathfrak{A})$, and by Theorem 1 we obtain $\mathrm{MdEq}^{\mathrm{reg}}(\mathfrak{A}) = \mathscr{H}\mathscr{S}\mathscr{P}\dot{\mathscr{E}}(\mathfrak{A}\cup\{\boldsymbol{1}\}) \subseteq \mathscr{H}\mathscr{S}\mathscr{S}_u\mathscr{P}(\mathfrak{A})$, which completes the proof of the theorem.

Remark. Note that Theorem 2 is proved without using the result (see [2]) that a class defined by regular equations is closed under formation of sums of direct systems, but using only the corresponding property of one-point extensions which are special cases of sums of direct systems. From this point of view Theorem 1 really shows that these special sums of direct systems are sufficient for characterizing the closure operator $\mathrm{MdEq}^{\mathrm{reg}}$ while the connections between the formation of sums of general direct systems and one-point extensions are described in Lemma 2.

REFERENCES

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