

ON THE DISTRIBUTION OF THE NUMBER OF SURVIVORS
AND DEATHS IN A BIRTH AND DEATH PROCESS

BY

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1. Introduction. In this paper, we consider pure birth and death processes and the problem is to find explicit formulae for the probability $p_{m,n}(t)$ that there are m survivors and n deaths at time t . Also the problem of evaluating the probability $p_{.,n}(t)$ that there are n deaths is considered.

In my previous paper [2], a procedure was presented to evaluate probabilities $p_{m,n}(t)$ when the birth rate $\lambda(t)$ and the death rate $\mu(t)$ are independent of time. This procedure consists in considering the system of differential equations for the partial generating functions

$$G_n(u, t) = \sum_{m=0}^{\infty} p_{m,n}(t) u^m.$$

Now we shall show that this method can be also applied when the ratio $\varrho = \mu(t)/\lambda(t)$ is independent of time. However, the exact formulae are complicated. Therefore approximate formulae for $p_{.,n}(t)$ that are applicable for ϱ close to unity or for large t are presented.

Steinberg and Stahl [3] have given formulae for $p_{.,n}(t)$ when $\lambda(t)$ and $\mu(t)$ are independent of time and ϱ is equal to 1. The first approximate formulae for $p_{.,n}(t)$, when ϱ is independent of time and close to unity, have been given by Gani and Yeo [1].

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2. Preliminaries. In this section, we shall consider the system

$$(1) \quad \frac{\partial G_n(u, t)}{\partial t} - (\lambda(t)u^2 - \beta(t)u) \frac{\partial G_n(u, t)}{\partial t} = \gamma(t) \frac{\partial G_{n-1}(u, t)}{\partial t},$$

$$n = 0, 1, \dots,$$

of partial differential equations coupled with the boundary conditions $G_0(u, 0) = u$, $G_1(u, 0) = 0$, $G_2(u, 0) = 0, \dots$. For $n < 0$ the function $G_n(u, t)$ is defined to be zero. Functions $\lambda(t)$, $\beta(t)$ and $\gamma(t)$ are assumed to be defined and continuous for $t > 0$. The function $\lambda(t)$ is supposed to be positive for $t > 0$. Throughout the paper, it is assumed that $\delta = \beta(t)/\lambda(t)$ is independent of time. Our problem is to find explicit formulae for $G_n(u, t)$ satisfying (1), where $n = 0, 1, \dots$

In the sequel, we shall consider the classes \mathfrak{A}_n of finite sequences defined by induction as follows:

$$1) \mathfrak{A}_1 = \{(2)\},$$

2) $\varepsilon^{(n)} = (\varepsilon_0^{(n)}, \dots, \varepsilon_{n-2}^{(n)}, 2)$, where $\varepsilon_i^{(n)}$ equal 0, 1 or 2, is contained k times in \mathfrak{A}_n if there are k sequences $\varepsilon^{(n-1)} = (\varepsilon_0^{(n-1)}, \dots, \varepsilon_{n-2}^{(n-1)})$ in \mathfrak{A}_{n-1} such that for each of them $(\varepsilon_0^{(n-1)} - \varepsilon_0^{(n)}, \dots, \varepsilon_{n-2}^{(n-1)} - \varepsilon_{n-2}^{(n)})$ is composed of $n-2$ elements equal to 0 and one element equal to 1.

The number of elements in the \mathfrak{A}_n 's may be evaluated by using the following evident recurrent relations:

If $a_k^{(n)}$ denotes the number of elements in \mathfrak{A}_n that include exactly k zeros, then

$$a_0^{(n)} = a_0^{(n-1)},$$

$$a_k^{(n)} = \begin{cases} (k+1)a_k^{(n-1)} + (n-2k)a_{k-1}^{(n-1)} & \text{for } k = 1, 2, \dots, n/2-1, \\ 0 & \text{for } k = n/2, n/2+1, \dots, n, \end{cases}$$

and

$$a_0^{(n)} = a_0^{(n-1)},$$

$$a_k^{(n)} = (k+1)a_k^{(n-1)} + (n-2k)a_{k-1}^{(n-1)} \quad \text{for } k = 1, 2, \dots, (n-3)/2,$$

$$a_{(n-1)/2}^{(n)} = a_{(n-3)/2}^{(n-1)},$$

$$a_k^{(n)} = 0 \quad \text{for } k = (n-1)/2+1, \dots, n,$$

for even and odd n , respectively.

Define

$$H(u, t_1, t_0) = \frac{U\delta}{\delta \exp\{\delta(\Lambda(t_1) - \Lambda(t_0))\} + [1 - \exp\{\delta(\Lambda(t_1) - \Lambda(t_0))\}]u},$$

where

$$\Lambda(t) = \int_0^t (x) dx.$$

Note that

$$(2) \quad H[H(u, t_2, t_1), t_1, t_0] = H(u, t_2, t_0),$$

$$(3) \quad \frac{\partial}{\partial u} H(u, t_1, t_0) = \left(\frac{H(u, t_1, t_0)}{u} \right)^2 \exp \{ \delta (\Lambda(t_1) - \Lambda(t_0)) \},$$

$$(4) \quad \frac{\partial}{\partial u} \left(\frac{H(u, t_1, 0)}{H(u, t_1, t_0)} \right) = \frac{1}{\delta} \left(\frac{H(u, t_1, 0)}{u} \right)^2 (1 - \exp \{ -\delta \Lambda(t_0) \}) \exp \{ \delta \Lambda(t_1) \}.$$

THEOREM 1. For the $G_n(u, t)$ satisfying (1), we have the formulae

$$(5) \quad G_0(u, t_0) = H(u, t_0, 0)$$

and, for $n \geq 1$,

$$(6) \quad G_n(u, t_n) = \frac{1}{\delta^{n-1}} \int_0^{t_n} \int_0^{t_{n-1}} \dots \int_0^{t_1} \prod_{i=0}^{n-1} \gamma(t_i) \exp \{ \delta \Lambda(t_i) \} \times \\ \times \sum_{\varepsilon^{(n)} \in \mathfrak{A}_n} 2^{n-b(\varepsilon^{(n)})} \prod_{i=0}^{n-1} \left(\frac{H(u, t_n, 0)}{H(u, t_n, t_i)} \right)^{\varepsilon_i^{(n)}} (1 - \exp \{ -\delta \Lambda(t_i) \})^{2-\varepsilon_i^{(n)}} dt_0 \dots dt_{n-1},$$

where $b(\varepsilon^{(n)})$ is the number of elements in $\varepsilon^{(n)}$ equal to 2.

Proof. If $n = 0$, then (1) reduces to

$$\frac{\partial G_0(u, t)}{\partial t} - (\lambda(t)u^2 - \beta(t)) \frac{\partial G_0(u, t)}{\partial u} = 0,$$

with $G_0(u, 0) = u$ as the boundary condition. In order to solve this equation first find the solution of

$$\frac{dt}{1} = - \frac{du}{\lambda(t)u^2 - \beta(t)u}.$$

Because $\delta = \beta(t)/\lambda(t)$ is independent of time, we obtain

$$(7) \quad u = \frac{\delta}{1 + c \exp \{ -\delta \Lambda(t) \}},$$

where c is a constant. Hence

$$G_0(u, t_0) = \frac{u\delta}{\delta \exp \{ \delta \Lambda(t_0) \} + (1 - \exp \{ \delta \Lambda(t_0) \})u}$$

or (5).

Now we proceed to prove (6) by induction. For $n = 1$, we can solve (1) in the standard manner to get

$$G_1(u, t) = \int_0^t \gamma(t_0) \left(\frac{\partial G_0(u, t_0)}{\partial u} \right) dt_0,$$

where

$$u = \delta(1 + c \exp\{-\delta\Lambda(t_0)\})^{-1}$$

and c has to be taken from (7).

Eliminating c and writing t_1 instead of t leads to

$$G_1(u, t_1) = \int_0^{t_1} \gamma(t_0) \left(\frac{\partial H(u, t_0, 0)}{\partial u} \right) dt_0$$

with $u = H(u, t_1, t_0)$. This, by (2) and (3), yields

$$G_1(u, t_1) = \int_0^{t_1} \gamma(t_0) \left(\frac{H(u, t_1, 0)}{H(u, t_1, t_0)} \right)^2 \exp\{\delta\Lambda(t_0)\} dt_0.$$

Thus (6) holds for $n = 1$.

Now we are going to prove that if formula (6) holds true for n , then it holds true for $n+1$. By applying for $n \geq 1$ the standard method, the solution of (1) is found to be

$$G_{n+1}(u, t_{n+1}) = \int_0^{t_{n+1}} \gamma(t_n) \left(\frac{\partial G_n(u, t_n)}{\partial u} \right) dt_n,$$

where $u = H(u, t_{n+1}, t_n)$. Since, by the inductive assumption, expression (6) holds true for n , this formula can be rewritten in the form

$$\begin{aligned} & G_{n+1}(u, t_{n+1}) \\ &= \frac{1}{\delta^{n-1}} \int_0^{t_{n+1}} \int_0^{t_n} \cdots \int_0^{t_1} \gamma(t_n) \prod_{i=0}^{n-1} \gamma(t_i) \exp\{\delta\Lambda(t_i)\} \times \\ & \times \sum_{\varepsilon^{(n)} \in \mathfrak{A}_n} 2^{n-b(\varepsilon^{(n)})} \frac{\partial}{\partial u} \left(\prod_{i=0}^{n-1} \left(\frac{H(u, t_n, 0)}{H(u, t_n, t_i)} \right)^{\varepsilon_i^{(n)}} (1 - \exp\{-\delta\Lambda(t_i)\})^{2-\varepsilon_i^{(n)}} \right) dt_0 \dots dt_n, \end{aligned}$$

where $u = H(u, t_{n+1}, t_n)$.

By definition of \mathfrak{A}_n and because of formulae (2)-(4) we get

$$G_{n+1}(u, t_{n+1}) = \frac{1}{\delta^n} \int_0^{t_{n+1}} \int_0^{t_n} \cdots \int_0^{t_1} \prod_{i=0}^n \gamma(t_i) \exp\{\delta \Lambda(t_i)\} \sum_{\epsilon^{(n+1)} \in \mathfrak{A}_{n+1}} 2^{n+1-b(\epsilon^{(n+1)})} \times \\ \times \prod_{i=0}^n \left(\frac{H(u, t_{n+1}, 0)}{H(u, t_{n+1}, t_i)} \right)^{\epsilon_i^{(n+1)}} (1 - \exp\{-\delta \Lambda(t_i)\})^{2-\epsilon_i^{(n+1)}} dt_0 dt_1 \dots dt_n.$$

This is the desired formula (6) for $n+1$.

COROLLARY 1. For every $n \geq 0$, there exists a system of $n+2$ functions $\psi_0^{(n)}(t), \psi_1^{(n)}(t), \dots, \psi_{n+1}^{(n)}(t)$ such that solution $G_n(u, t)$ of (1) can be presented as

$$(8) \quad G_n(u, t) = \frac{1}{\delta^{n+1}} \frac{\psi_0^{(n)}(t) + \frac{1}{\delta} \psi_1^{(n)}(t)u + \dots + \frac{1}{\delta^{n+1}} \psi_{n+1}^{(n)}(t)u^{n+1}}{\left[1 - \frac{1}{\delta} (1 - \exp\{-\delta \Lambda(t)\})u\right]^{n+1}}.$$

Proof. For $n = 0$, we have

$$G_0(u, t_0) = \frac{u\delta}{\delta \exp\{\delta \Lambda(t_0)\} + (1 - \exp\{\delta \Lambda(t_0)\})u}$$

or

$$G_0(u, t) = \frac{1}{\delta^{-1}} \frac{\psi_0^{(0)}(t) + \frac{1}{\delta} \psi_1^{(0)}(t)u}{1 - \frac{1}{\delta} (1 - \exp\{-\delta \Lambda(t)\})u},$$

where $\psi_0^{(0)}(t) \equiv 0$ and $\psi_1^{(0)}(t) = \exp\{-\delta \Lambda(t)\}$. Thus (8) holds for $n = 0$. For $n \geq 1$ formula (8) is a consequence of (6). In fact, since

$$\frac{H(u, t_1, 0)}{H(u, t_1, t_0)} = \frac{1 - \frac{1}{\delta} (1 - \exp\{-\delta(\Lambda(t_1) - \Lambda(t_0))\})u}{1 - \frac{1}{\delta} (1 - \exp\{-\delta \Lambda(t_1)\})u} e^{-\delta(t_0)},$$

formula (6) reduces to

$$G_n(u, t_n) = \frac{1}{\delta^{n+1} \left[1 - \frac{1}{\delta} (1 - \exp\{-\delta \Lambda(t_n)\})u\right]^{n+1}} \int_0^{t_n} \int_0^{t_{n-1}} \cdots \int_0^{t_1} \prod_{i=0}^{n-1} \gamma(t_i) e^{\delta \Lambda(t_i)} \times \\ \times \sum_{\epsilon^{(n)} \in \mathfrak{A}_n} 2^{n-b(\epsilon^{(n)})} \prod_{i=0}^{n-1} \left[1 - \frac{1}{\delta} (1 - e^{-\delta(\Lambda(t_n) - \Lambda(t_i))})u\right]^{\epsilon_i^{(n)}} e^{-\epsilon_i^{(n)} \delta \Lambda(t_i)} \times \\ \times (1 - e^{-\delta \Lambda(t_i)})^{2-\epsilon_i^{(n)}} dt_0 \dots dt_{n-1}.$$

Whatever is the sequence $\varepsilon^{(n)} \in \mathfrak{A}_n$, the expression under the integrals is a polynomial of degree $n+1$ with respect to u . Thus this formula can be presented as (8).

COROLLARY 2. *The coefficients $q_{m,n}(t)$ of the expansion of $G_n(u, t)$ can be presented in the form*

$$(9) \quad q_{m,n}(t) = \frac{1}{\delta^{m+n-1}} (1 - \exp\{-\delta A(t)\})^m \sum_{i=0}^j \binom{m+n-i}{n} \frac{\psi_i^{(n)}(t)}{(1 - \exp\{-\delta A(t)\})^i},$$

where $j = \min(m, n+1)$.

Proof. This can be readily deduced from (8).

Remark 1. We would like to note that this procedure can be applied to obtain the formulae for $q_{m_0, m_1, \dots, m_k}(t)$ arising in the expansion of the solution $G(u_0, \dots, u_k, t)$ of

$$\frac{\partial G}{\partial t} - (\lambda(t)u_0^2 - \beta(t)u_0) \frac{\partial G}{\partial u_0} = \frac{\partial G}{\partial u_0} \sum_{i=1}^r \gamma_i(t) u_i, \quad r \geq 1,$$

with $G(u_0, \dots, u_k, 0) = u_0$ and also in the expansion of the solution $G(u_0, \dots, u_k, t)$ of

$$\frac{\partial G}{\partial t} - (\lambda(t)u_0^2 - \beta(t)u_0) \frac{\partial G}{\partial u_0} = \sum_{i=1}^r \gamma_i(t) u_i^2 \frac{\partial^2 G}{\partial u_0 \partial u_i}, \quad r \geq 1,$$

with $G(u_0, \dots, u_k, 0) = u_0$.

3. The exact and approximate formulae. Let us now consider a pure birth and death process with birth rate $\lambda(t)$ and death rate $\mu(t)$ and one alive individual at time zero.

Let

$$G(u, v, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{m,n}(t) u^m v^n$$

be the probability generating function of the considered process. It is known that

$$\frac{\partial G}{\partial t} - [\lambda(t)u^2 - (\lambda(t) + \mu(t))u + \mu(t)v] \frac{\partial G}{\partial u} = 0$$

and $G(u, v, 0) = u$. When $\varrho = \mu(t)/\lambda(t)$ is independent of time (this will be assumed throughout the paper) the solution is (see [1])

$$G(u, v, t) = \frac{(\eta + \vartheta(v))(u - \eta + \vartheta(v)) - (\eta - \vartheta(v))(u - \eta - \vartheta(v))e^{2A(t)\vartheta(v)}}{u - \eta + \vartheta(v) - (u - \eta - \vartheta(v))e^{2A(t)\vartheta(v)}},$$

where $\vartheta(v) = \frac{1}{2}[(1 + \varrho)^2 - 4\varrho v]^{1/2}$ and $\eta = (1 + \varrho)/2$.

Because the explicit formulae for the involved probabilities $p_{m,n}(t)$ and $p_{.,n}(t)$ cannot be obtained from this formula, we propose (see [2]) to use theorem 1.

The differential equation for the generating function $G(u, v, t)$ can be split into a system of recurrent partial differential equations

$$(10) \quad \frac{\partial G_n}{\partial t} - [\lambda(t)u^2 - (\lambda(t) + \mu(t))u] \frac{\partial G_n}{\partial u} = \mu(t) \frac{\partial G_{n-1}}{\partial u}, \quad n = 0, 1, \dots,$$

coupled with the boundary conditions $G_0(u, 0) = u$, $G_1(u, 0) = 0$, $G_2(u, 0) = 0, \dots$, where

$$G_n(u, t) = \sum_{m=0}^{\infty} p_{m,n}(t) u^m.$$

System (10) is the system (1) with $\beta(t) = \lambda(t) + \mu(t)$ and $\gamma(t) = \mu(t)$. Formulae (5) and (6) reduce to

$$(11) \quad G_0(u, t) = (1 + \varrho) \frac{\frac{1}{\alpha} \frac{1 + \varrho}{1 + \varrho} u}{1 - \frac{1 - \alpha}{1 + \varrho} u}$$

and

$$(12) \quad G_n(u, t) = \frac{\varrho^n}{(1 + \varrho)^{n-1}} \frac{1}{\left(1 - \frac{1 - \alpha}{1 + \varrho} u\right)^{n+1}} \int_0^t \int_0^{t_{n-1}} \dots \int_0^{t_1} \prod_{i=0}^{n-1} \lambda(t_i) e^{(1+\varrho)A(t_i)} \times \\ \times \sum_{\varepsilon^{(n)} \in \mathfrak{A}_n} 2^{n-b(\varepsilon^{(n)})} \prod_{i=0}^{n-1} \left[1 - \frac{1}{1 + \varrho} (1 - \alpha e^{(1+\varrho)A(t_i)}) u\right]^{\varepsilon_i^{(n)}} \times \\ \times e^{-(1+\varrho)\varepsilon_i^{(n)}} (1 - e^{(1+\varrho)A(t_i)})^{2-\varepsilon_i^{(n)}} dt_0 \dots dt_{n-1},$$

respectively, where $\alpha = \exp\{-(1 + \varrho)A(t)\}$. Introducing the variables $\xi_i = \exp\{(1 + \varrho)A(t_i)\}$, $i = 0, 1, \dots, n-1$, leads to

$$(13) \quad G_n(u, t) = \frac{\varrho^n}{(1 + \varrho)^{2n-1}} \frac{1}{\left(1 - \frac{1 - \alpha}{1 + \varrho} u\right)^{n+1}} \int_1^{\alpha^{-1}\xi_{n-1}} \int_1^{\xi_1} \dots \int_1^{\xi_1} \sum_{\varepsilon^{(n)} \in \mathfrak{A}_n} 2^{n-b(\varepsilon^{(n)})} \times \\ \times \prod_{i=0}^{n-1} \left[\frac{1}{\xi_i} - \frac{1}{1 + \varrho} \left(\frac{1}{\xi_i} - \alpha\right) u\right]^{\varepsilon_i^{(n)}} \left(1 - \frac{1}{\xi_i}\right)^{2-\varepsilon_i^{(n)}} d\xi_0 \dots d\xi_{n-1}.$$

Thus from corollaries 1 and 2, we get for $m, n = 0, 1, \dots$ the formulae

$$p_{m,n}(t) = \frac{\varrho^n}{(1+\varrho)^{2n+m-1}} \sum_{i=0}^j \binom{m+n-i}{n} (1-a)^{m-i} \Phi_i^{(n)}(t),$$

where $j = \min(m, n+1)$ and

$$p_{.,n}(t) = \frac{\varrho^n}{(1+\varrho)^{2n-1}} \frac{1}{(\varrho+a)^{n+1}} \sum_{i=0}^{n+1} (1+\varrho)^{n+1-i} \Phi_i^{(n)}(t).$$

$\Phi_0^{(n)}(t), \dots, \Phi_{n+1}^{(n)}(t)$ can be evaluated by integration from (13). Unfortunately, the evaluation of these functions is extremely tedious.

In particular, for $n = 0, 1, 2$, we get

$$\begin{aligned} \Phi_0^{(0)} &= 0, \\ \Phi_1^{(0)} &= a; \\ \Phi_0^{(1)} &= 1-a, \\ \Phi_1^{(1)} &= -2(1-a) - 2a \ln a, \\ \Phi_2^{(1)} &= 1-a^2 + 2a \ln a; \\ \Phi_0^{(2)} &= (1-a^2) + 2a \ln a, \\ \Phi_1^{(2)} &= -3(1-a^2) - 2a(2+a) \ln a + 2a \ln^2 a, \\ \Phi_2^{(2)} &= 3(1-a)^2 + 6a^2 \ln a - 2a(2+a) \ln^2 a, \\ \Phi_3^{(2)} &= -(1-a)^2(1+a) - 2a(1-a) \ln a - 2a(1+a) \ln^2 a \end{aligned}$$

and

$$\begin{aligned} p_{.,0}(t) &= (1+\varrho) \frac{a}{\varrho+a}, \\ p_{.,1}(t) &= \frac{\varrho}{1+\varrho} \frac{(1-a)(\varrho^2+a) - 2a\varrho \ln a}{(\varrho+a)^2}, \\ p_{.,2}(t) &= \frac{\varrho^2}{(1+\varrho)^3} \times \\ &\times \frac{(1-a^2)(\varrho^3+a) - 2a[\varrho^3+a - \varrho(1-\varrho)(1-a)] \ln a + 2\varrho a(\varrho-a) \ln^2 a}{(\varrho+a)^3}. \end{aligned}$$

Explicit general formulae for the $p_{.,n}(t)$'s are known only for $\varrho = 1$ (see [3]), namely,

$$(14) \quad p_{.,0}(t) = 1 - 8\Lambda(t) \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 \pi^2 + 4\Lambda^2(t)}$$

and, for $n \geq 1$,

$$(15) \quad p_{.,n}(t) = 2^{2n+1} A^{2n-1}(t) \sum_{k=0}^{\infty} \frac{(2k+1)^2 \pi^2}{[(2k+1)^2 \pi^2 + 4A^2(t)]^{n+1}}.$$

Now we proceed to derive approximate formulae for $p_{.,n}(t)$ that are applicable for ϱ close to unity or for large t .

THEOREM 2. *If $\alpha/\varrho < 1$, then*

$$(16) \quad p_{.,0}(t) = \frac{1}{2}(1+\varrho) \left(1 - 4\Theta \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 \pi^2 + \Theta^2} \right),$$

and, for $n \geq 1$,

$$(17) \quad p_{.,n}(t) = \frac{2^{2n+1} \varrho^n}{(1+\varrho)^{2n-1}} \Theta^{2n+1} \sum_{k=0}^{\infty} \frac{(2k+1)^2 \pi^2}{[(2k+1)^2 \pi^2 + \Theta^2]^{n+1}} + O[(1-\varrho) \alpha \ln^{n-1} \alpha],$$

where $\Theta = (1+\varrho)A(t) + \ln \varrho$.

Proof. If $\varrho = 1$, then (14) coincide with (11) and (15) with (12). Hence for $0 < \alpha < 1$, we have

$$(18) \quad \frac{2\alpha}{1+\alpha} = 1 + 4 \ln \alpha \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 \pi^2 + \ln^2 \alpha}$$

and

$$(19) \quad \int_1^{\alpha^{-1}} \int_1^{\xi_{n-1}} \dots \int_1^{\xi_1} \sum_{\epsilon^{(n)} \in \mathfrak{U}_n} 2^{n-b(\epsilon^{(n)})} \left(\frac{1}{\xi_i} + \alpha \right)^{\epsilon_i^{(n)}} \left(1 - \frac{1}{\xi_i} \right)^{2-\epsilon_i^{(n)}} d\xi_0 \dots d\xi_{n-1} \\ = 2^{2n+1} (1+\alpha)^{n+1} \ln^{2n-1} \frac{1}{\alpha} \sum_{k=0}^{\infty} \frac{(2k+1)^2 \pi^2}{[(2k+1)^2 \pi^2 + \ln^2 \alpha]^{n+1}}, \quad n = 1, 2, \dots$$

Formula (16) follows from (18) by substituting α/ϱ instead of α . In order to prove (17) split integral (13) into two parts

$$(20) \quad G_n(1, t) = \frac{\varrho^n}{(1+\varrho)^{2n-1}} \frac{1}{(1+\alpha/\varrho)^{n+1}} \int_1^{\varrho \alpha^{-1}} \int_1^{\xi_{n-1}} \dots \int_1^{\xi_1} \sum_{\epsilon^{(n)} \in \mathfrak{U}_n} 2^{n-b(\epsilon^{(n)})} \times \\ \times \prod_{i=0}^{n-1} \left(\frac{1}{\xi_i} + \frac{\alpha}{\varrho} \right)^{\epsilon_i^{(n)}} \left(1 - \frac{1}{\xi_i} \right)^{2-\epsilon_i^{(n)}} d\xi_0 \dots d\xi_{n-1} + \frac{\varrho^n}{(1+\varrho)^{2n-1}} \frac{1}{(1+\alpha/\varrho)^{n+1}} \times \\ \times \int_{\varrho \alpha^{-1}}^{\alpha^{-1}} \int_1^{\xi_{n-1}} \dots \int_1^{\xi_1} \sum_{\epsilon^{(n)} \in \mathfrak{U}_n} 2^{n-b(\epsilon^{(n)})} \times \prod_{i=0}^{n-1} \left(\frac{1}{\xi_i} + \frac{\alpha}{\varrho} \right)^{\epsilon_i^{(n)}} \left(1 - \frac{1}{\xi_i} \right)^{2-\epsilon_i^{(n)}} d\xi_0 \dots d\xi_{n-1}.$$

Using (15) with a/ϱ instead of a leads to

$$p_{.,n}(t) = \frac{\varrho^n}{(1+\varrho)^{2n-1}} 2^{2n+1} \ln^{2n-1} \frac{\varrho}{\alpha} \sum_{k=0}^{\infty} \frac{(2k+1)^2 \pi^2}{[(2k+1)^2 \pi^2 + \ln^2 \alpha / \varrho]^{n+1}} + R_n(a, \varrho),$$

where $R_n(a, \varrho)$ denotes the second term in the right-hand side of (20).

We will prove (17) by showing that

$$(21) \quad R_n(a, \varrho) = O[(1-\varrho) \ln^{n-1} a].$$

The functions that correspond to the particular sequences $\epsilon^{(n)} \mathfrak{A}_n$ in $R_n(a, \varrho)$ under the integrals are sums of expressions of the form

$$(22) \quad \frac{\alpha^L}{\xi_{i_1}^{\beta_{i_1}} \dots \xi_{i_s}^{\beta_{i_s}}},$$

where β_{i_l} equal 1 or 2 and $0 \leq s \leq n$, multiplied by some constants. In order to prove (21) we shall find the minimum value of L for a fixed sequence $(\beta_{i_1}, \dots, \beta_{i_s})$.

Take any sequence $\epsilon^{(n)} \mathfrak{A}_n$. Let n_0, n_1, n_2 be the number of elements in $\epsilon^{(n)}$ equal to 0, 1 and 2, respectively. Note that we can write $n_0 = k$, $n_1 = n-1-2k$ and $n_2 = k+1$ with k within limits $0 \leq k \leq (n-1)/2$. Let r_1 and r_2 be the number of 1's and 2's in $(\beta_{i_1}, \dots, \beta_{i_s})$, and m_0, m_1 and m_2 the number of 0's, 1's and 2's in the subsequence $(\epsilon_{i_1}^{(n)}, \dots, \epsilon_{i_s}^{(n)})$ of $\epsilon^{(n)}$, respectively. Define $r = r_1 + 2r_2$. Finally, let r_{ij} be the number of pairs $(\epsilon_l^{(n)}, \epsilon_l^{(n)})$, $l = i_1, \dots, i_s$, that are equal to (i, j) , where $i = 0, 1, 2$ and $j = 1, 2$. Then $r_j = r_{0j} + r_{1j} + r_{2j}$ for $j = 1, 2$.

It is clear that for the considered sequences $(\beta_{i_1}, \dots, \beta_{i_s})$ and $\epsilon^{(n)}$ the power L in (22) is equal to

$$n-1-2k-m_1+2(k+1-m_2)+r_{21}.$$

Since $m_0+m_1+m_2=s$ and $r_{21}+r_{22}=m_2$, we get

$$L = n+1-(s-m_0)-r_{22}.$$

Thus L depends on $\epsilon^{(n)}$ and $(\beta_{i_1}, \dots, \beta_{i_s})$ only through m_0 and r_{22} . Since the smallest possible value of m_0 is 0, and the greatest value of r_{22} is attained when $r_{02} = r_{12} = 0$, or equivalently, when $r_{22} = r_2$, and since $r_2 = s-r$, we get

$$\inf L = n+1-r.$$

Relation (21) follows now from the formula

$$\int_{\alpha^{-1}}^{\alpha^{-1} \xi_{n-1}} \int_1^{\xi_1} \dots \int_1^{\xi_1} \frac{1}{\xi_{i_1}^{\beta_{i_1}} \dots \xi_{i_s}^{\beta_{i_s}}} d\xi_0 \dots d\xi_{n-1} = O \left[(1-\varrho) \frac{\ln^{n-1} a}{\alpha^{n-r}} \right],$$

where, as previously, $r = \sum_{j=1}^s \beta_{i_j}$.

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