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ASSOCIATE AND PSEUDOASSOCIATE SETS IN LCA GROUPS, II

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1. Introduction. Let G be a compact Abelian group with dual group Γ. A compact subset K of G and a subset Λ of Γ are *associated* (respectively, *pseudoassociated*) if for each $\varphi \in B(\Lambda)$ (respectively, $\varphi \in l^\infty(\Lambda)$) there exists a measure μ (respectively, a pseudomeasure S) concentrated on K and such that $\hat{\mu}|_\Lambda = \varphi$ (respectively, $\hat{S}|_\Lambda = \varphi$). Professor S. Hartman has asked whether there is a relationship between the notions of associate and pseudoassociate. In a previous note [1] we proved the existence of pairs K, Λ that were pseudoassociate, but not associate. In this note we construct a pair K, Λ that is associate, but not pseudoassociate.

We now describe the construction. We work on the circle group T. Let K be a Kronecker set, and let Λ be a union of arithmetic progressions of increasing length. In the next section we show that K and Λ may be constructed in such a way that K and Λ are associate. Let us now point out why such K and Λ can never be pseudoassociate.

Since Λ has arbitrarily long arithmetic progressions, $B(\Lambda) \neq l^\infty(\Lambda)$, as is well known. Since K is a Kronecker set, $M(K) = PM(K)$ by a result of Varopoulos (see [2]). Hence

$$PM(K)^\wedge|_\Lambda = M(K)^\wedge|_\Lambda \subseteq B(\Lambda) \neq l^\infty(\Lambda).$$

A necessary and sufficient condition for K and Λ to be associate is that there exists a number $\delta > 0$ such that, for all trigonometric polynomials f with frequencies in Λ,

$$\delta \|f\|_\infty \leqslant \sup\{|f(x)|:\ x \in K\},$$

but our construction can be simplified only slightly using this fact.

2. The construction. Let $\lambda(1) = 0$, and let $x \in T$ be any element of infinite order. Let $E(1)$ be any finite set of integers such that, for each $z \in T$, there exists $\gamma \in E(1)$ with $|z - \langle \gamma, x\rangle| < 2^{-11}$. Let $\delta(1) = 2^{-11}$. Then for each probability measure ν on T there exists a probability measure σ on $K(1) = \{x\}$ such that $\hat{\nu} = \hat{\sigma}$ on $\lambda(1)$. Let $U(1)$ be a compact neighborhood of $0 \in T$ such that $|1-\langle\gamma, u\rangle| < 2^{-13}$ for $u \in U(1)$ and $\gamma \in E(1)$. This begins our inductive construction.

Suppose that $n \geqslant 1$, and that non-negative integers $\lambda(1), \lambda(2), \ldots, \lambda(n)$, finite subsets $E(1) \subseteq E(2) \subseteq \ldots \subseteq E(n)$ of integers, independent finite subsets $K(1) \subseteq K(2) \subseteq \ldots \subseteq K(n)$ of T, compact neighborhoods $U(1)$, $U(2), \ldots, U(n)$ of $0 \in T$ and numbers $\delta(1) > \delta(2) > \ldots > \delta(n) > 0$ have been found so that the following conditions hold:

(1) $$j\lambda(j) < \lambda(j+1) \quad \text{for } 1 \leqslant j < n;$$

(2) $$\lambda(j+1)\,U(j) = T \quad \text{for } 1 \leqslant j < n;$$

(3) $$\delta(j) + |1-\langle k\lambda(j+1), x\rangle| < 2^{-j-9} \quad \text{for } x \in K(j) \text{ and } 1 \leqslant k \leqslant j < n;$$

(4) for $1 \leqslant j \leqslant n$ and each probability measure ν on T, there exists a probability measure σ on $K(j)$ such that

$$\delta(j) + \sum_{l=1}^{k} \left|\hat{\nu}\big(l\lambda(k)\big) - \hat{\sigma}\big(l\lambda(k)\big)\right| < 2^{-k-8} \quad \text{for } 1 \leqslant k \leqslant j;$$

(5) $$|1-\langle k\lambda(j+1), x\rangle| < \delta(j)/2 \quad \text{for } x \in K(j) \text{ and } 1 \leqslant k \leqslant j < n;$$

(6) for $x, y \in K(j)$, $x + U(j) \cap y + U(j) \neq \emptyset$ implies $x = y$ for $1 \leqslant j \leqslant n$;

(7) $$K(j) + U(j) \supseteq K(j+1) \quad \text{for } 1 \leqslant j < n;$$

(8) $$\operatorname{Int} U(j) \supseteq U(j+1) + U(j+2) + \ldots + U(l) \quad \text{for } 1 \leqslant j < l \leqslant n;$$

(9) for each $f\colon K(j) \to T$, there exists $\gamma \in E(j)$ such that

$$\delta(n) + |f(x) - \langle\gamma, x\rangle| < 2^{-j-9} \quad \text{for } x \in K(j) \text{ and } 1 \leqslant j \leqslant n;$$

(10) $$|1-\langle x, \gamma\rangle| < \delta(j)/3j \quad \text{for } x \in U(j),\ \gamma \in E(j) \text{ and } 1 \leqslant j \leqslant n;$$

(11) $$|1-\langle x, j\lambda(k)\rangle| < \delta(k)/3k \quad \text{for } x \in U(k) \text{ and } 1 \leqslant j \leqslant k \leqslant n.$$

We set $\Lambda(j) = \{k\lambda(j)\colon 1 \leqslant k \leqslant j\}$ for $1 \leqslant j \leqslant n$.

We now produce $\lambda(n+1)$ and $K(n+1)$. Let $N \geqslant n\lambda(n)$ be so large that $NU(n) = T$. Let $\lambda = \lambda(n+1) \geqslant N$ be such that

(12) $$|1-\langle k\lambda, x\rangle| < \delta(n)/3n \quad \text{for } x \in K(n) \text{ and } 1 \leqslant k \leqslant n+1.$$

Let $\Lambda(n+1) = \{k\lambda\colon 1 \leqslant k \leqslant n+1\}$. Then, for each probability measure ν on T, there exists a probability measure σ_2 on $U(n)$ such that

(13) $$\sum_{\gamma \in \Lambda(n+1)} |\hat{\nu}(\gamma) - \hat{\sigma}_2(\gamma)| < 2^{-n-7}.$$

Let $F = K(n)+U(n)$. A straightforward argument shows that there exists a finite set $K \subseteq F$ such that for each probability measure σ_2 on F there is a probability measure σ_3 on K such that

$$\sum_{\gamma\in\Lambda(k)} |\hat{\sigma}_2(\gamma)-\hat{\sigma}_3(\gamma)| < \min\big(2^{-k-7}, \delta(n)/3\big) \quad \text{for } 1 \leqslant k \leqslant n+1, \tag{14}$$

and such that $K(n+1) = K(n)\cup K$ is independent.

Now let ν be any probability measure on T. Then there exists a probability measure σ_1 on $K(n)$ such that (4) holds for $1 \leqslant k \leqslant n$, and there exists a probability measure σ_2 on $U(n)$ such that (13) holds. Then (4) and (11) imply

$$2\delta(n)/3+\sum_{\gamma\in\Lambda(k)} |\hat{\nu}(\gamma)-\hat{\sigma}_1(\gamma)\hat{\sigma}_2(\gamma)| < 2^{-k-9} \qquad \text{for } 1\leqslant k\leqslant n \tag{15}$$

while (12) and (13) imply

$$2\delta(n)/3+\sum_{\gamma\in\Lambda(k)} |\hat{\nu}(\gamma)-\hat{\sigma}_1(\gamma)\hat{\sigma}_2(\gamma)| < 2^{-n-10}. \tag{16}$$

By our choice of K and (15) there exists a probability measure σ_4 on K such that

$$\delta(n)/3+\sum_{\gamma\in\Lambda(k)} |\hat{\nu}(\gamma)-\hat{\sigma}_4(\gamma)| < 2^{-k-9} \qquad \text{for } 1\leqslant k\leqslant n+1. \tag{17}$$

This proves that $K(n+1)$, $\lambda(n+1)$ and $\Lambda(n+1)$ have properties (1)-(5) if $0 < \delta(n+1) \leqslant \delta(n)/3$.

The construction of $E(n+1)$ is straightforward, as is the choice of $U(n+1)$.

The induction is complete.

We let

$$K = \bigcap_{n=1}^{\infty} \big(K(n)+U(n)\big).$$

That K is a Kronecker set follows from (7)-(10), exactly as in [3], p. 101 and 102. Let

$$\Lambda = \bigcup_{k=1}^{\infty} \Lambda(k).$$

We now show that $M(K)\hat{}\,|_\Lambda = B(\Lambda)$. Let $\varphi \in B(\Lambda)$. Then there exist non-negative measures $\mu_1, \mu_2, \ldots, \mu_4$ such that, on Λ,

$$\varphi = \hat{\mu}_1-\hat{\mu}_2+i\hat{\mu}_3-i\hat{\mu}_4 \quad \text{and} \quad 2\|\varphi\| \geqslant \sum_{j=1}^{4} \|\mu_j\|.$$

Let $\sigma_1^{(n)}, \sigma_2^{(n)}, \ldots, \sigma_4^{(n)}$ be probability measures on $K(n)$ such that (4) holds for $\nu_j = \|\mu_j\|^{-1}\mu_j$. Let

$$\Lambda_n = \bigcup_{j=1}^{n} \Lambda(j),$$

and

$$\omega_n = \|\mu_1\|\sigma_1^{(n)} + \|\mu_2\|\sigma_2^{(n)} + i\|\mu_3\|\sigma_3^{(n)} - i\|\mu_4\|\sigma_4^{(n)}.$$

Then (4) implies

$$\|\varphi|_{\Lambda_n} - \hat{\omega}_n|_{\Lambda_n}\|_{B(\Lambda_n)} \leqslant 4\sum_{k=1}^{n} 2^{-k-8}2\|\varphi\| = 2^{-3}\|\varphi\|_{B(\Lambda_n)}.$$

Taking weak-* limits, we see that, for each $\varphi \in B(\Lambda)$, there exists $\sigma \in M(K)$ such that

$$\|\sigma\| \leqslant 2\|\varphi\|_{B(\Lambda)} \quad \text{and} \quad \|\hat{\sigma}|_{\Lambda} - \varphi\|_{B(\Lambda)} < 2^{-3}\|\varphi\|_{B(\Lambda)}. \tag{18}$$

A standard iteration using (18) shows that $M(K)\hat{}\,|_{\Lambda} = B(\Lambda)$.
The construction is complete.

REFERENCES

[1] C. C. Graham, *Associate and pseudoassociate sets in LCA groups*, Colloquium Mathematicum 38 (1977), p. 97-101.

[2] L. A. Lindal and F. Poulsen, *Thin sets in harmonic analysis*, New York 1972.

[3] W. Rudin, *Fourier analysis on groups*, New York 1962.

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