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ON SOME PROPERTIES OF ERDÖS SETS

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1. With every Hamel basis H of the space $\boldsymbol{R}^N$ we can associate the set H^* of all finite combinations of elements of H with integral coefficients. The set H^* will be referred to as the *Erdös set associated with* H, since Erdös [2] first considered such sets in the one-dimensional case ($N = 1$).

Let $D \subset \boldsymbol{R}^N$ be an open and convex set. A function $f\colon D \to \boldsymbol{R}$ is called *convex* whenever the relation

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}$$

holds for all $x, y \in D$. A function $f\colon \boldsymbol{R}^N \to \boldsymbol{R}$ is called *additive* whenever the relation

$$f(x+y) = f(x)+f(y) \tag{1}$$

holds for all $x, y \in \boldsymbol{R}^N$.

In the sequel we will need the following definitions (cf. [4] and [7]).

A set $T \subset \boldsymbol{R}^N$ *belongs to the class* $\mathscr{A}$ whenever if $D \subset \boldsymbol{R}^N$ is an open and convex set such that $T \subset D$ and $f\colon D \to \boldsymbol{R}$ is a convex function bounded from above on T, then f is continuous.

A set $T \subset \boldsymbol{R}^N$ *belongs to the class* $\mathscr{B}$ whenever every additive function $f\colon \boldsymbol{R}^N \to \boldsymbol{R}$ bounded from above on T is continuous.

A set $T \subset \boldsymbol{R}^N$ *belongs to the class* $\mathscr{A}_c$ whenever if $D \subset \boldsymbol{R}^N$ is an open and convex set such that $T \subset D$ and $f\colon D \to \boldsymbol{R}$ is a convex function such that the restriction $f|T$ is continuous, then f is continuous in D.

A set $T \subset \boldsymbol{R}^N$ *belongs to the class* $\mathscr{B}_c$ whenever every additive function $f\colon \boldsymbol{R}^N \to \boldsymbol{R}$, such that the restriction $f|T$ is continuous, is continuous in $\boldsymbol{R}^N$.

M. E. Kuczma [14] proved that

$$\mathscr{A} = \mathscr{B} \tag{2}$$

whereas for classes $\mathscr{A}_c$ and $\mathscr{B}_c$ we have only the inclusion $\mathscr{A}_c \subset \mathscr{B}_c$ (cf. [7]). We have also the inclusion (cf. [7]) $\mathscr{A}_c \subset \mathscr{A}$.

A set $A \subset \boldsymbol{R}^N$ is called *saturated non-measurable* (cf. [5]) if

$$m_i(A) = m_i(\boldsymbol{R}^N \setminus A) = 0, \tag{3}$$

where m_i stands for the N-dimensional inner Lebesgue measure. The N-dimensional outer Lebesgue measure will be denoted by m_e.

It follows directly from (3) that if $A \subset \boldsymbol{R}^N$ is a saturated non-measurable set and $E \subset \boldsymbol{R}^N$ is a Lebesgue measurable set of positive measure, then $A \cap E \neq \emptyset$.

In establishing that a set is saturated non-measurable, the following Smital's lemma (cf. [13], [11]) is often helpful:

LEMMA 1. *Let $B, D \subset \boldsymbol{R}^N$ be such that $m_e(B) > 0$ and D is dense in $\boldsymbol{R}^N$. If*

$$A = B + D = \{x \in \boldsymbol{R}^N \mid x = b + d,\ b \in B,\ d \in D\},$$

then $m_i(\boldsymbol{R}^N \setminus A) = 0$.

In the sequel the dimension N of the underlying space is regarded as fixed.

2. In [9] we proved that, in the one-dimensional case ($N = 1$), if $H \subset \boldsymbol{R}$ is an arbitrary Hamel basis and $I \subset \boldsymbol{R}$ is an arbitrary interval, then $I \cap H^* \in \mathscr{A}$. Now we are going to extend this result to arbitrary N.

THEOREM 1. *If $H \subset \boldsymbol{R}^N$ is an arbitrary Hamel basis and $A \subset \boldsymbol{R}^N$ is an arbitrary set with non-empty interior, then*

$$A \cap H^* \in \mathscr{A}. \tag{4}$$

Proof. First we assume that

$$0 \in \operatorname{int} A. \tag{5}$$

Since H spans $\boldsymbol{R}^N$ over the rationals, it also spans $\boldsymbol{R}^N$ over the reals and, consequently, it contains a basis $\{b_1, \ldots, b_N\}$ of $\boldsymbol{R}^N$ over $\boldsymbol{R}$:

$$b_i \in H \subset H^*, \quad i = 1, \ldots, N. \tag{6}$$

Every $x \in \boldsymbol{R}^N$ can be uniquely represented as

$$x = \lambda_1 b_1 + \ldots + \lambda_N b_N \tag{7}$$

with real $\lambda_1, \ldots, \lambda_N$.

Let $f\colon \boldsymbol{R}^N \to \boldsymbol{R}$ be an additive function and assume that f is discontinuous. Put

$$\varphi_i(\lambda) = f(\lambda b_i), \quad i = 1, \ldots, N;\ \lambda \in \boldsymbol{R}. \tag{8}$$

For x of the form (7), by (1) we have

$$f(x) = \varphi_1(\lambda_1) + \ldots + \varphi_N(\lambda_N).$$

If every function $\varphi_i: \boldsymbol{R} \to \boldsymbol{R}$, $i = 1, \ldots, N$, were continuous, f would also be continuous. Consequently, there exists an i_0 such that φ_{i_0} is discontinuous. In the sequel we suppress the index i_0 and write φ instead of φ_{i_0} and b instead of b_{i_0}. Note that according to (8) and (1) the function $\varphi: \boldsymbol{R} \to \boldsymbol{R}$ is additive.

Since φ is discontinuous, there exists a λ_0 such that $\varphi(\lambda_0) \neq \lambda_0 \varphi(1)$, say

$$\varphi(\lambda_0) = \lambda_0 \varphi(1) + c, \quad c \neq 0. \tag{9}$$

It follows from (9) and from the rational homogeneity of every additive function (cf. [1]) that λ_0 is an irrational number.

The element $\lambda_0 b$ of $\boldsymbol{R}^N$ has a representation

$$\lambda_0 b = \mu_1 h_1 + \ldots + \mu_m h_m,$$

where $h_1, \ldots, h_m \in H$ and $\mu_1, \ldots, \mu_m$ are rational. Consequently, there exists a positive integer k such that all the numbers $k\mu_1, \ldots, k\mu_m$ are integers, whence

$$k\lambda_0 b \in H^*. \tag{10}$$

Let $\boldsymbol{Z}$ denote the set of integers, and $\boldsymbol{N}$ the set of positive integers. The set

$$D = \{\alpha \in \boldsymbol{R} \mid \alpha = p + qk\lambda_0,\ p, q \in \boldsymbol{Z}\}$$

is dense in $\boldsymbol{R}$. Moreover, for every $\alpha \in D$, $\alpha = p + qk\lambda_0$, by (6) and (10) we have

$$\alpha b = pb + qk\lambda_0 b \in H^*.$$

We may choose a sequence $\{\alpha_n\}$ of points of D such that

$$\alpha_n \neq 0 \quad \text{for } n \in \boldsymbol{N}, \tag{11}$$

$$\lim_{n \to \infty} \alpha_n = 0. \tag{12}$$

Moreover, we have

$$\alpha_n b \in H^* \quad \text{for } n \in \boldsymbol{N}. \tag{13}$$

Let $\alpha_n = p_n + q_n k\lambda_0$, $n \in \boldsymbol{N}$. It follows from (11) and (12) that the sequence $\{q_n\}$ is unbounded. Replacing, if necessary, the sequence $\{\alpha_n\}$ by a suitable subsequence, and α_n by $-\alpha_n$ (which does not spoil conditions (12) and (13)) we may assume that

$$\lim_{n \to \infty} ckq_n = \infty. \tag{14}$$

Now, according to (9) we have

$$\begin{aligned} f(\alpha_n b) = \varphi(\alpha_n) &= \varphi(p_n + q_n k\lambda_0) = \varphi(p_n) + \varphi(q_n k\lambda_0) \\ &= p_n \varphi(1) + q_n k\varphi(\lambda_0) = p_n \varphi(1) + q_n k(\lambda_0 \varphi(1) + c) \\ &= \varphi(1)\,[p_n + q_n k\lambda_0] + q_n kc = \alpha_n \varphi(1) + ckq_n, \end{aligned}$$

whence by (12) and (14) we obtain

$$\lim_{n \to \infty} f(\alpha_n b) = \infty. \tag{15}$$

According to (5) and (12) we have $\alpha_n b \in A$ for large n, whence by (13) we get

$$\alpha_n b \in A \cap H^* \quad \text{for } n > n_0.$$

Consequently, relation (15) shows that f cannot be bounded on $A \cap H^*$. Thus $A \cap H^* \in \mathscr{B}$. Hence, by (2), we obtain (4).

Before proceeding further with the proof let us note the following

LEMMA 2. *If $H \subset \boldsymbol{R}^N$ is an arbitrary Hamel basis, then the set H^* is dense in $\boldsymbol{R}^N$.*

To see this let $\{b_1, \ldots, b_N\} \subset H$ be a basis of $\boldsymbol{R}^N$ over $\boldsymbol{R}$. The argument above (where λ_0 may be an arbitrary irrational number) shows that the set H^* is dense on every line $\boldsymbol{R}b_i$, $i = 1, \ldots, N$. Hence it follows easily, since the sum of elements of H^* is again in H^*, that H^* is dense in $\boldsymbol{R}^N$.

Now we complete the proof of Theorem 1. Let us drop assumption (5). By Lemma 2 there exists an $h \in H^* \cap \operatorname{int} A$. For the set $A-h$ we have $0 \in \operatorname{int}(A-h)$, whence, by what has already been proved, $(A-h) \cap H^* \in \mathscr{A}$. Hence it follows (cf. [10]) that

$$[(A-h) \cap H^*] + h \in \mathscr{A}.$$

But $[(A-h) \cap H^*] + h = [(A-h)+h] \cap [H^* + h] = A \cap H^*$, and (4) holds.

Hence, taking $A = \boldsymbol{R}^N$ we obtain

COROLLARY 1. *If $H \subset \boldsymbol{R}^N$ is an arbitrary Hamel basis, then $H^* \in \mathscr{A}$.*

In the one-dimensional case this result was first established by Ger [3].

As a consequence of Theorem 1 we have also the following

THEOREM 2. *If $H \subset \boldsymbol{R}^N$ is an arbitrary Hamel basis and $A \subset \boldsymbol{R}^N$ is an arbitrary set with non-empty interior, then*

$$A \cap H^* \in \mathscr{A}_c. \tag{16}$$

Proof. Let $D \subset \boldsymbol{R}^N$ be an arbitrary open and convex set such that $A \cap H^* \subset D$ and let $f\colon D \to \boldsymbol{R}$ be an arbitrary convex function such that the restriction $f|A \cap H^*$ is continuous. By Lemma 1 there exists an $h \in (\operatorname{int} A) \cap H^*$. Since $f|A \cap H^*$ is continuous at h, there exists a neighbourhood U of h such that $f|A \cap H^*$ is bounded on $U \cap A \cap H^*$. Clearly, we have $\operatorname{int}(U \cap A) \neq \emptyset$, whence, by Theorem 1, $U \cap A \cap H^* \in \mathscr{A}$. Consequently, the function f, being – in particular – bounded from above on $U \cap A \cap H^*$, is continuous. Hence we obtain (16).

COROLLARY 2. *If $H \subset \boldsymbol{R}^N$ is an arbitrary Hamel basis, then $H^* \in \mathscr{A}_c$.*

3. In the one-dimensional case ($N = 1$) Erdös proved [2] that for every Hamel basis H the associated set H^* is saturated non-measurable. This is true for arbitrary N and to prove this we use essentially the same argument as in [2].

THEOREM 3. *If $H \subset \mathbf{R}^N$ is an arbitrary Hamel basis, then the set H^* is saturated non-measurable.*

Proof. Clearly, we have

$$\mathbf{R}^N = \bigcup_{n=1}^{\infty} \frac{1}{n} H^*,$$

whence

(17) $$m_e(H^*) > 0.$$

On the other hand, we have

(18) $$H^* = H^* + H^*,$$

whence, by Lemmas 1 and 2, $m_i(\mathbf{R}^N \setminus H^*) = 0$.

Finally, if we had $m_i(H^*) > 0$, then relation (18), in view of the theorem of Steinhaus (e.g., [6], [15], and [16]), would imply that $\operatorname{int} H^* \neq \emptyset$, which is impossible. Consequently,

(19) $$m_i(H^*) = 0,$$

and the theorem holds.

We will also prove the following

LEMMA 3. *If $H \subset \mathbf{R}^N$ is an arbitrary Hamel basis and $f\colon \mathbf{R}^N \to \mathbf{R}$ is an arbitrary discontinuous additive function, then for every $k \in \mathbf{R}$ the set*

(20) $$A_k = \{x \in H^* \mid f(x) > k\}$$

is saturated non-measurable.

Proof. There exists an $x_0 \in H^*$ such that $d = f(x_0) > 0$. For $n \in \mathbf{Z}$ put

$$B_n = \{x \in H^* \mid nd \leqslant f(x) < (n+1)d\}.$$

We will show that

(21) $$B_n + x_0 = B_{n+1} \quad \text{for } n \in \mathbf{Z}.$$

Take an $x \in B_n + x_0$. This means that $x = t + x_0$, where $t \in B_n$. Hence

(22) $$nd \leqslant f(t) < (n+1)d$$

and, since $f(x) = f(t) + f(x_0) = f(t) + d$, we get

(23) $$(n+1)d \leqslant f(x) < (n+2)d.$$

Further, since $t \in H^*$ and $x_0 \in H^*$, by (18) we have $x \in H^*$. Consequently, $x \in B_{n+1}$, which shows that $B_n + x_0 \subset B_{n+1}$. Conversely, if $x \in B_{n+1}$, then

$x \in H^*$ and $f(x)$ fulfils (23). Hence $t = x - x_0 \in H^*$, and $f(t) = f(x) - f(x_0) = f(x) - d$ fulfils (22). Thus $t \in B_n$ and $x = t + x_0 \in B_n + x_0$. Consequently, $B_{n+1} \subset B_n + x_0$, which completes the proof of (21).

Relation (21) shows that all sets B_n, $n \in \boldsymbol{Z}$, are congruent under translation, and hence they have the same outer measure. Since

$$\bigcup_{n=-\infty}^{+\infty} B_n = H^*,$$

according to (17) we must have

$$m_e(B_n) > 0 \quad \text{for } n \in \boldsymbol{Z}. \tag{24}$$

Since for every $k \in \boldsymbol{R}$ there exists an $n \in \boldsymbol{Z}$ such that $B_n \subset A_k$, relation (24) implies

$$m_e(A_k) > 0 \quad \text{for } k \in \boldsymbol{R}. \tag{25}$$

Now, suppose that for an open set G we have $A_k \cap G = \emptyset$ for a $k \in \boldsymbol{R}$. Then $f \leqslant k$ on $G \cap H^*$ and, by Theorem 1, f would be continuous. This shows that every set A_k, $k \in \boldsymbol{R}$, is dense in $\boldsymbol{R}^N$.

Fix a $k \in \boldsymbol{R}$ and take $x, y \in A_{k/2}$. Then $x, y \in H^*$, whence by (18) we obtain $x + y \in H^*$, and $f(x) > \frac{1}{2}k$, $f(y) > \frac{1}{2}k$, which implies $f(x+y) = f(x) + f(y) > k$. Consequently, $x + y \in A_k$. This shows that $A_{k/2} + A_{k/2} \subset A_k$. By Lemma 1, relation (25), and the fact that $A_{k/2}$ is dense in $\boldsymbol{R}^N$, we have

$$m_i(\boldsymbol{R}^N \setminus A_k) \leqslant m_i[\boldsymbol{R}^N \setminus (A_{k/2} + A_{k/2})] = 0.$$

On the other hand, since $A_k \subset H^*$, by (19) we have $m_i(A_k) = 0$. Consequently, the set (20) is saturated non-measurable, which was to be proved.

4. Now we can improve on theorems of Section 2.

THEOREM 4. *If $H \subset \boldsymbol{R}^N$ is an arbitrary Hamel basis and $A \subset \boldsymbol{R}^N$ is an arbitrary Lebesgue measurable set of positive measure, then* (4) *holds.*

Proof. Let $f\colon \boldsymbol{R}^N \to \boldsymbol{R}$ be an arbitrary discontinuous additive function and let the set A_k, $k \in \boldsymbol{R}$, be defined by (20). By Lemma 3 we have $A \cap A_k \neq \emptyset$ for every $k \in \boldsymbol{R}$, which means that f is not bounded from above on $A \cap H^*$. Consequently, $A \cap H^* \in \mathscr{B}$, which in view of (2) implies (4).

THEOREM 5. *If $H \subset \boldsymbol{R}^N$ is an arbitrary Hamel basis and $A \subset \boldsymbol{R}^N$ is an arbitrary Lebesgue measurable set of positive measure, then* (16) *holds.*

Proof. Let $D \subset \boldsymbol{R}^N$ be an arbitrary open and convex set such that $A \cap H^* \subset D$ and let $f\colon D \to \boldsymbol{R}$ be an arbitrary convex function such that the restriction $f|A \cap H^*$ is continuous. By the Lebesgue density theorem, the set of those points of A which are density points of A has positive measure, and thus, by Theorem 3, has a non-void intersection with H^*. Let $h \in A \cap H^*$ be a density point of A. Since $f|A \cap H^*$ is continuous at h, there exists a neighbourhood U of h such that $f|A\cap H^*$ is bounded on $U\cap A\cap H^*$. The set $U\cap A$ has positive measure, so by Theorem 4 we get $U\cap A\cap H^*\in\mathcal{A}$. Consequently, the function f, being — in particular — bounded from above on $U\cap A\cap H^*$, is continuous. Hence we obtain (16).

5. By quite similar arguments one can derive topological analogues of the results in Sections 3 and 4.

A set $A\subset \mathbf{R}^N$ is called *saturated non-Baire* (cf. [12]) if neither A nor $\mathbf{R}^N\setminus A$ contains a second category Baire set. (In this definition the space $\mathbf{R}^N$ can be replaced by an arbitrary topological space X.)

THEOREM 6. *If $H\subset \mathbf{R}^N$ is an arbitrary Hamel basis, then the set H^* is saturated non-Baire.*

LEMMA 4. *If $H\subset \mathbf{R}^N$ is an arbitrary Hamel basis and $f\colon \mathbf{R}^N\to \mathbf{R}$ is an arbitrary discontinuous additive function, then for every $k\in \mathbf{R}$ the set* (20) *is saturated non-Baire.*

THEOREM 7. *If $H\subset \mathbf{R}^N$ is an arbitrary Hamel basis and $A\subset \mathbf{R}^N$ is an arbitrary second category Baire set, then* (4) *holds.*

THEOREM 8. *If $H\subset \mathbf{R}^N$ is an arbitrary Hamel basis and $A\subset \mathbf{R}^N$ is an arbitrary second category Baire set, then* (16) *holds.*

The proofs of Theorems 6-8 and of Lemma 4 do not differ essentially from those of Theorems 3-5 and of Lemma 3, and therefore are not given here. However, let us note that, instead of Lemma 1, its topological analogue must be used. Such an analogue may be found in [12].

From Theorems 5 and 8 we obtain the following results which in the one-dimensional case ($N = 1$) have been proved in [7] and [8].

COROLLARY 3. *If $A\subset \mathbf{R}^N$ is an arbitrary Lebesgue measurable set of positive measure, then $A\in\mathcal{A}_c$.*

COROLLARY 4. *If $A\subset \mathbf{R}^N$ is an arbitrary second category Baire set, then $A\in\mathcal{A}_c$.*

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