COLLOQUIUM MATHEMATICUM

VOL. XL 1979 FASC. 2

DENSITY CHARACTER, BARRELLEDNESS AND THE CLOSED GRAPH THEOREM

BY

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1. Introduction. In [7], Kalton described those locally convex spaces which can serve as domain spaces for a closed graph theorem in which the range space is an arbitrary separable Banach space. Subsequently, in [10] we considered a similar problem where the range space is now an arbitrary Banach space of linear dimension at most $\mathfrak{c}$, the cardinal number of the real line. It is not difficult to see that a Banach space has linear dimension at most $\mathfrak{c}$ if and only if its density character is at most $\mathfrak{c}$. The main purpose of this note is to examine the general situation where we have Banach spaces of arbitrarily prescribed maximum density character a. We describe the corresponding domain spaces, which we call *$G(a)$-barrelled spaces*, and investigate their basic properties. We also relate them to the a-barrelled spaces of Valdivia [15].

For simplicity, we will restrict attention here to separated locally convex spaces, although our results extend easily to the non-separated case. If E is a locally convex space, then E' will represent its (continuous) dual and E^* its algebraic dual. We shall use both the forms X and (X, ξ) to represent a topological space, the latter being employed when it is convenient to name the topology ξ on the underlying set X. The restriction of a topology ξ and a mapping f to a subset Y will be written as $\xi|_Y$ and $f|_Y$, respectively. $|A|$ will stand for the cardinality of a set A. Generally, we follow the topological vector-space notation of [12].

We are indebted to Dr. A. G. Hamilton for some helpful discussions on cardinal arithmetic. The first-named author gratefully acknowledges the support of a Commonwealth Academic Staff Scholarship.

2. Some barrelledness concepts. Let (E, ξ) be a locally convex space, let B be a barrel in (E, ξ) and let a be an infinite cardinal number. If

$$q: E \to E/\bigcap_{\lambda>0} \lambda B$$

is the quotient map, the *gauge* of $q(B)$ is a norm on $E/\bigcap_{\lambda>0}\lambda B$. We shall say that B is a $G(\alpha)$-*barrel* in (E, ξ) if $E/\bigcap_{\lambda>0}\lambda B$ has density character at most α for the resulting norm topology, and that (E, ξ) is $G(\alpha)$-*barrelled* if each $G(\alpha)$-barrel in (E, ξ) is a ξ-neighbourhood of 0. Clearly, if α and β are infinite cardinal numbers with $\alpha < \beta$, then each $G(\alpha)$-barrel is a $G(\beta)$-barrel and each $G(\beta)$-barrelled space is $G(\alpha)$-barrelled. Also, a locally convex space is barrelled if and only if it is $G(\alpha)$-barrelled for each infinite cardinal α.

Our first result gives a dual characterization of $G(\alpha)$-barrels.

THEOREM 1. *Let E be a locally convex space and let A be a non-empty absolutely convex $\sigma(E', E)$-bounded set. Then the following are equivalent:*

(a) A° *is a $G(\alpha)$-barrel;*

(b) *the uniformity induced on A by the $\sigma(E', E)$-uniformity on E' has a base consisting of at most α sets;*

(c) $\sigma(E', E)|_A$ *has a base of neighbourhoods of* 0 *consisting of at most α sets.*

Proof. Let C be the $\sigma(E^*, E)$-closure of A and let H be the linear span of C. Then H is the dual of the normed space $E/\bigcap_{\lambda>0}\lambda A^\circ$, and C is the closed unit ball of H.

Suppose that (a) holds. Let D be a norm-dense subset of $E/\bigcap_{\lambda>0}\lambda A^\circ$ with $|D| \leqslant \alpha$ and let Φ be the set of non-empty finite subsets of D. We note that $\sigma(E^*, E)$ and $\sigma\big(H, E/\bigcap_{\lambda>0}\lambda A^\circ\big)$ coincide on C, that D separates the elements of H, and that a separated compact space has a unique uniformity. Consequently,

$$\mathscr{B} = \big\{\{(x', y') \in C \times C\colon\ |\langle x, x' - y'\rangle| \leqslant r,\, x \in \varphi\}\colon\ r \in Q^+,\, \varphi \in \Phi\big\}$$

is a base for the uniformity induced on C by the $\sigma(E', E)$-uniformity on E'; it is easily seen that $|\mathscr{B}| \leqslant \aleph_0\alpha = \alpha$. This gives us (b).

Trivially, (b) $\Rightarrow$ (c).

Suppose finally that (c) holds. We can find a set $\{\varphi_\lambda : \lambda \in \Lambda\}$ of non-empty finite subsets of E such that $|\Lambda| \leqslant \alpha$ and

$$\big\{\{x' \in A\colon\ |\langle x, x'\rangle| \leqslant 1,\, x \in \varphi_\lambda\}\colon\ \lambda \in \Lambda\big\}$$

is a base of neighbourhoods of 0 for $\sigma(E', E)|_A$. For each non-empty finite subset φ of E, the set

$$\{x' \in A\colon |\langle x, x'\rangle| \leqslant 1,\, x \in \varphi\}$$

is $\sigma(E^*, E)$-dense in $\{x' \in C\colon |\langle x, x'\rangle| \leqslant 1,\, x \in \varphi\}$. Consequently,

$$\big\{\{x' \in C\colon |\langle x, x'\rangle| \leqslant 1,\, x \in \varphi_\lambda\}\colon \lambda \in \Lambda\big\}$$

is a base of neighbourhoods of 0 for $\sigma(E^*, E)|_C$. If $x' \in C\setminus\{0\}$, we can therefore find $x \in \bigcup\{\varphi_\lambda: \lambda \in \Lambda\}$ such that $|\langle x, x'\rangle| > 1$. This implies that the set of equivalence classes in $E/\bigcap_{\lambda>0}\lambda A^\circ$ of the elements of $\bigcup\{\varphi_\lambda: \lambda \in \Lambda\}$ is total for the norm topology. Taking rational or complex-rational linear combinations gives us a norm-dense subset of cardinality at most a. This establishes (a).

In the notation of the proof, $\sigma(E^*, E)|_C$ is the coarsest topology on C making each $x|_C$ continuous ($x \in D$). Since the scalar field has a countable base for its topology, we deduce

COROLLARY. *Let B be a $G(a)$-barrel in a locally convex space E. Then $\sigma(E', E)|_{B^\circ}$ has a base consisting of at most a sets and each subset of B° has a $\sigma(E', E)$-dense subset of cardinality at most a.*

Remarks. 1. For an infinite cardinal a, Valdivia [15] calls a locally convex space E *a-barrelled* if each $\sigma(E', E)$-bounded set of cardinality at most a is equicontinuous. An $\aleph_0$-barrelled space is also said to be *ω-barrelled* [9] or *σ-barrelled* [2]. We infer from the Corollary that each a-barrelled space is $G(a)$-barrelled, but — as we shall see in the sequel (Examples (ii)-(iv)) — the reverse implication does not hold in general.

2. Let E be a locally convex space. The proof of Theorem 1 effectively shows that the $G(\aleph_0)$-barrels are precisely the polars of the absolutely convex $\sigma(E', E)$-bounded metrizable sets. Consequently, the $G(\aleph_0)$-barrelled spaces are the elements of Kalton's class $\mathscr{C}(\zeta) = \mathscr{C}(\zeta_B)$ ([7], Theorem 2.6).

Theorem 1 and the Corollary to Theorem 3 of [11] show that the $G(\mathfrak{c})$-barrels are the polars of the (absolutely convex) $\sigma(E', E)$-bounded essentially separable sets and that the $G(\mathfrak{c})$-barrelled spaces coincide with the δ-barrelled spaces (see [10] or [11] for definitions).

We now consider intersections of $G(a)$-barrels.

THEOREM 2. *Let E be a locally convex space and let $\mathscr{B}$ be a set of $G(a)$-barrels in E such that*

$$B_* = \bigcap\{B: B \in \mathscr{B}\}$$

is absorbent. If $a^{|\mathscr{B}|} = a$, then B_* is a $G(a)$-barrel in E.

Proof. Let σ be an ordinal whose cardinal is $|\mathscr{B}|$ and list the elements of $\mathscr{B}$ as B_λ ($\lambda \in [0, \sigma)$). For each $\lambda \in [0, \sigma)$ we can find a subset D_λ of E such that

(a) given $\varepsilon > 0$ and $x \in E$, there exists a $y \in D_\lambda$ such that

$$|\langle x-y, x'\rangle| \leqslant \varepsilon \quad \text{for all } x' \in B_\lambda^\circ,$$

(b) $|D_\lambda| \leqslant a$.

Let F be the Banach space of bounded scalar-valuet functions on $C = \bigcup\{B_\lambda^\circ: \lambda \in [0, \sigma)\}$ with the supremum norm. Let $x \in E$ and $\varepsilon > 0$.

For each $\lambda \in [0, \sigma)$, choose $y = y_\lambda$ so that (a) holds, and define f on C by

$$f(x') = \begin{cases} \langle y_0, x' \rangle & \text{if } x' \in B_0^\circ, \\ \langle y_\lambda, x' \rangle & \text{if } x' \in B_\lambda^\circ \setminus \bigcup \{B_\mu^\circ \colon \mu \in [0, \lambda)\}, \lambda \in (0, \sigma). \end{cases}$$

We have

$$(*) \qquad |\langle x, x' \rangle - f(x')| \leqslant \varepsilon \quad \text{for all } x' \in C,$$

and

$$(**) \qquad \sup\{|f(x')| \colon x' \in C\} \leqslant \varepsilon + \sup\{|\langle x, x' \rangle| \colon x' \in C\} < \infty.$$

Let G be the collection of all functions f constructed in this way for all $x \in E$ and $\varepsilon > 0$. Then, by (**), $G \subseteq F$ and, by (*), $\{x|_C \colon x \in E\}$ is contained in the closure of G in F. Also $|G| \leqslant a^{|\mathscr{B}|} = a$. Since F is a metric space, we can now deduce that $\{x|_C \colon x \in E\}$ has a uniformly dense subset of cardinality at most a. It now follows that $B_* = C^\circ$ is a $G(a)$-barrel.

COROLLARY 1. *Any non-zero finite intersection of $G(a)$-barrels is a $G(a)$-barrel.*

Proof. If $B_1, B_2, \ldots, B_n$ are $G(a)$-barrels, then $\bigcap_{r=1}^{n} B_r$ is absorbent and $a^n = a$. The result now follows from Theorem 2.

Corollary 1 shows that the set of $G(a)$-barrels in a locally convex space (E, ξ) forms a base of neighbourhoods of 0 for a locally convex topology, η say, on E (see [12], Chapter I, Theorem 2). Since $\{x \in E \colon |\langle x, x' \rangle| \leqslant 1\}$ is trivially a $G(a)$-barrel for each $x' \in E'$, we certainly infer that η is finer than $\sigma(E, E')$. If (E, ξ) is $G(a)$-barrelled, then η is easily seen to be the coarsest $G(a)$-barrelled topology of the dual pair (E, E'). If (E, ξ) is $G(\mathfrak{c})$-barrelled, then η is the topology $\delta(E, E')$ of [10], Theorem 2.

COROLLARY 2. *Let E be a $G(a)$-barrelled space. If $a^{\aleph_0} = a$, then E is countably barrelled* [5] *under the coarsest $G(a)$-barrelled topology of the dual pair (E, E').*

This follows from Theorem 1 of [5] by taking a countable $\mathscr{B}$ in our Theorem 2.

COROLLARY 3. *Let E be a locally convex space and let a, γ be infinite cardinals such that $2^\gamma \leqslant a$. If A is a non-empty $\sigma(E', E)$-bounded set with $|A| \leqslant \gamma$, then A° is a $G(a)$-barrel. Consequently, if E is $G(a)$-barrelled, then it is also γ-barrelled.*

Proof. As previously, $\{x \in E \colon |\langle x, x' \rangle| \leqslant 1\}$ is a $G(2^\gamma)$-barrel for each $x' \in E'$ and, in view of $(2^\gamma)^\gamma = 2^{\gamma\gamma} = 2^\gamma$, it follows from Theorem 2 that

$$A^\circ = \bigcap_{x' \in A} \{x \in E \colon |\langle x, x' \rangle| \leqslant 1\}$$

is a $G(2^\gamma)$-barrel since A° is absorbent. As already observed, this implies that A° is a $G(a)$-barrel. The second assertion is now immediate.

Remarks. 1. Theorem 2 holds, in particular, if

$$a = 2^\beta \quad \text{and} \quad |\mathscr{B}| \leqslant \beta \quad \text{for } a \leqslant a^{|\mathscr{B}|} \leqslant a^\beta = (2^\beta)^\beta = 2^\beta = a.$$

We may then take $|\mathscr{B}| \leqslant \aleph_0$ so that Corollary 2 holds for any such a.

In Example (iii) we will show that Corollary 2 can fail if $\aleph_0 \leqslant a < \mathfrak{c}$. Assuming the Generalized Continuum Hypothesis (GCH), in Example (iv) we construct counterexamples to Theorem 2 and Corollary 2 for all cases where the cardinality assumption does not hold.

2. Even if $a = 2^\beta$, then a $G(a)$-barrelled space E may fail to be countably barrelled if its topology is not the coarsest $G(a)$-barrelled topology of the dual pair (E, E') (see [11], Section 3).

3. If we accept GCH, we infer from Corollary 3 that a $G(a)$-barrelled space is γ-barrelled for any infinite cardinal γ such that $\gamma < a$. However, as shown in Example (iii), we cannot reach this conclusion without some further axiom.

3. Examples. In the following, $K = \boldsymbol{R}$ or $\boldsymbol{C}$ and for each non-negative real number r we put $I_r = \{x \in K: |x| \leqslant r\}$.

(i) Let a and β be infinite cardinals with $a < \beta$. Let B be any set of cardinality β and put

$$E = K^{(B)}, \quad E' = \{x' \in K^B: |\operatorname{supp} x'| \leqslant a\},$$

where $\operatorname{supp} x' = \{\lambda \in B: \xi_\lambda \neq 0\}$ if $x' = (\xi_\lambda)_{\lambda \in B} \in K^B$. Each $\sigma(E', E)$-bounded set of cardinality at most a is contained in a subset of E' of the form $\prod\{I_{r(\lambda)}: \lambda \in B\}$, where $|\{\lambda: r(\lambda) \neq 0\}| \leqslant a$. Consequently, $(E, \tau(E, E'))$ is a-barrelled and, therefore, $G(a)$-barrelled. However, $(E, \tau(E, E'))$ is not $G(\beta)$-barrelled, since $I_1^B \cap E'$ is not $\sigma(E', E)$-relatively compact while its polar is a $G(\beta)$-barrel (cf. [15], Theorem 2).

(ii) Let A be an infinite set with $|A| = a \geqslant \mathfrak{c}$. Identifying $l_\infty(A)$ with the space of bounded continuous scalar-valued functions on the discrete space A, we see that the set of unit point masses δ_x for $x \in \beta A$ provides 2^{2^a} distinct elements of $l_\infty(A)'$. Since $\|\delta_x - \delta_y\| = 2$ if $x \neq y$, $\{\delta_x: x \in \beta A\}$ has no norm-dense proper subset and, consequently, the Banach space $l_\infty(A)'$ must have density character (at least) 2^{2^a}. Therefore, the closed unit ball B of $l_\infty(A)'$ is not a neighbourhood of zero if $l_\infty(A)'$ has the coarsest $G(2^a)$-barrelled topology η of the dual pair $(l_\infty(A)', l_\infty(A)'')$.

The closed unit ball C of $l_\infty(A)$ is $\sigma(l_\infty(A)'', l_\infty(A)')$-dense in B° and $|C| = \mathfrak{c}^a = 2^a$. Thus, under η, $l_\infty(A)'$ is $G(2^a)$-barrelled but not 2^a-barrelled (cf. [11], the Remark following the Corollary to Theorem 3).

(iii) Suppose that $\aleph_0 \leqslant a < \mathfrak{c}$ and let η be the coarsest $G(a)$-barrelled topology of the dual pair $(K^{(K)}, K^K)$. Now $(I_1^K)^\circ$ is a barrel but not a $G(a)$-

barrel in $K^{(K)}$. To see this, note that if D is any subset of $K^{(K)}$ of cardinality at most a, then

$$\left|\bigcup\{\operatorname{supp} x\colon x \in D\}\right| \leqslant a\aleph_0 = a.$$

Then for any $\lambda \in K \setminus \bigcup\{\operatorname{supp} x\colon x \in D\}$ we have

$$x' = (\delta_{\mu\lambda})_{\mu\in K} \in I_1^K \quad \text{and} \quad \langle x, x'\rangle = 0 \quad \text{for all } x \in D.$$

Since I_1^K is separable under $\sigma(K^K, K^{(K)})$ (see [4]), $K^{(K)}$ is neither σ-barrelled nor countably barrelled under η.

(iv) Here we assume GCH. Let a be any infinite cardinal and let Λ be any set of cardinality 2^a. By [4], I_1^Λ has a $\sigma(K^\Lambda, K^{(\Lambda)})$-dense subset of cardinality a, say $\{x_\nu\colon \nu \in N\}$, where $|N| = a$. If β_0 is the cofinality of a, then we can find a family $\{N_\mu\}_{\mu\in M}$ of non-empty subsets of N such that

$$\bigcup\{N_\mu\colon \mu \in M\} = N, \quad |N_\mu| < a \ \ (\mu \in M) \quad \text{and} \quad |M| = \beta_0.$$

For each $\mu \in M$ let $A_\mu = \{x_\nu\colon \nu \in N_\mu\}$. Then $2^{|A_\mu|} \leqslant a$ so that each A_μ° is a $G(a)$-barrel by Corollary 3. As in (iii), $(I_1^\Lambda)^\circ = \bigcap\{A_\mu^\circ\colon \mu \in M\}$ is a barrel but not a $G(a)$-barrel.

Let E be a locally convex space with scalar field K and suppose that there is a set $\mathscr{B}$ of $G(a)$-barrels in E whose intersection is absorbent and such that $|\mathscr{B}| \geqslant \beta_0$. Then

$$\mathscr{B}' = \{A_\mu^\circ \times B\colon \mu \in M, B \in \mathscr{B}\}$$

is a set of $G(a)$-barrels in $K^{(\Lambda)} \times E$ of the same cardinality. The intersection of the elements of $\mathscr{B}'$ is a barrel but not a $G(a)$-barrel. From [3], Theorem 7.3, we note that $a^\beta = a$ if and only if $\beta < \beta_0$. Consequently, Theorem 2 can fail in each case where the hypothesis on the cardinals does not hold.

In particular, we deduce that if a has cofinality $\aleph_0$, then $K^{(\Lambda)}$ is neither a-barrelled nor countably barrelled under the coarsest $G(a)$-barrelled topology of the dual pair $(K^{(\Lambda)}, K^\Lambda)$.

4. Closed graph theorems and permanence properties. We begin by characterizing $G(a)$-barrelled spaces by means of a closed graph theorem. First we require

LEMMA. *Let E and F be locally convex spaces and let $t\colon E \to F$ be a linear mapping. If B is a $G(a)$-barrel in F, then* $\operatorname{cl} t^{-1}(B)$ *is a $G(a)$-barrel in E.*

Proof. Let

$$q\colon F \to F/\bigcap_{\lambda>0} \lambda B$$

be the quotient map. We can find a subset D of E such that $|D| \leqslant a$ and $q(t(D))$ is norm-dense in $q(t(E))$. Given $x \in E$ and $\varepsilon > 0$, we can therefore find $y \in D$ such that $t(x)-t(y) \in \varepsilon B$, which implies

$$x-y \in \varepsilon t^{-1}(B) \subseteq \varepsilon \operatorname{cl} t^{-1}(B).$$

The result now follows.

THEOREM 3. *Let E be a locally convex space. Then E is $G(a)$-barrelled if and only if, whenever F is a Banach space of density character at most a and $t: E \to F$ is a linear mapping with a closed graph, t is continuous.*

Proof. The necessity of the condition follows immediately from the Lemma and from [8], 11.1, since the closed unit ball of a Banach space of density character at most a is trivially a $G(a)$-barrel.

The proof of the sufficiency is standard. For a $G(a)$-barrel B in E, let F be the completion of the normed space $E/\bigcap_{\lambda>0} \lambda B$. Then F is a Banach space of density character at most a, and the quotient map

$$q: E \to E/\bigcap_{\lambda>0} \lambda B$$

has a closed graph in $E \times F$. Consequently, if the condition is satisfied, then q is continuous as a mapping of E into F, which implies that B is a neighbourhood of 0 in E. (See, for example, the proof of Proposition 11, Chapter VI in [12].)

Now, the following permanence properties of $G(a)$-barrelled spaces can immediately be obtained by applying Theorems 2.1 and 2.2 of [6] and the methods of Corollary 1 to Theorem 3 in [10].

COROLLARY. (a) *An inductive limit of $G(a)$-barrelled spaces is $G(a)$-barrelled.*

(b) *Any product of $G(a)$-barrelled spaces is $G(a)$-barrelled.*

(c) *If E is a $G(a)$-barrelled space and H is any subspace of the completion of E which contains E, then H is also $G(a)$-barrelled.*

Remark. We note that, by Remark 2 following the Corollary to Theorem 1, in the cases $a = \aleph_0$ and $a = \mathfrak{c}$ Theorem 3 gives us the equivalence of (iii) and (iv) in Theorem 2.6 of [7] and Theorem 3 of [10].

In [10], Theorem 4, we show that the δ-barrelled spaces have the countable codimensional subspace property. In fact, this holds for all $G(a)$-barrelled spaces.

THEOREM 4. *Let (E, ξ) be a $G(a)$-barrelled space and let F be a countable codimensional subspace of E. Then $(F, \xi|_F)$ is $G(a)$-barrelled.*

Proof. Let B be a $G(a)$-barrel in F and let H be the closure of F in E. Let A be the set of extensions by continuity of the elements of B° (polar in F') to H. Since E is also $G(\aleph_0)$-barrelled, we deduce from the Corollary to Theorem 1.4 and Theorem 2.6 in [7] that E' is $\sigma(E', E)$-sequentially complete. Since F and H must have countable codimension in H and E, respectively, it follows from the Theorem in Section 4 and the Lemma in Section 2 of [9] that A is an absolutely convex $\sigma(H', H)$-bounded set. Let L be a supplement for H in E and extend each $x' \in A$ to an element of E^* by putting $\langle x, x'\rangle = 0$ for each $x \in L$. By the Lemma of Section 2 in [13], the resulting set C of extensions is contained in E'; it is clearly $\sigma(E', E)$-bounded and absolutely convex.

By Theorem 1 there is a family $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ of non-empty finite subsets of F such that $|\Lambda| \leqslant a$ and

$$\big\{\{x' \in B^\circ : |\langle x, x'\rangle| \leqslant 1, x \in \varphi_\lambda\} : \lambda \in \Lambda\big\}$$

is a base of neighbourhoods of 0 for $\sigma(F', F)|_{B^\circ}$. Let $\{y_n : n \in N\}$ be a subset of $(H\setminus F)\cup\{0\}$ which spans a supplement M of F in H. By expressing each element of E as the sum of elements of F, M and L, it is easily seen that the sets

$$\{x' \in C : |\langle x, x'\rangle| \leqslant 1, x \in \varphi_\lambda\} \cap \left\{x' \in C : |\langle y_r, x'\rangle| \leqslant \frac{1}{n}, r = 1, 2, \ldots, n\right\} \qquad (\lambda \in \Lambda, n \in N)$$

form a base $\mathscr{B}$ of neighbourhoods of 0 for $\sigma(E', E)|_C$. Since $|\mathscr{B}| \leqslant a\aleph_0 = a$, we infer from Theorem 1 that C° is a $G(a)$-barrel in E. The result now follows since $C^\circ \cap F = B$.

Finally, we give a closed graph theorem characterizing a-barrelled spaces. This is not so satisfactory as Theorem 3 since it involves a "mixed" condition.

THEOREM 5. *Let (E, ξ) be a locally convex space and let a be an infinite cardinal number. Then the following are equivalent:*

(i) *(E, ξ) is a-barrelled.*

(ii) *Let $t: E \to F$ be a linear mapping such that*

(a) *F is the dual of a Banach space H with density character at most a,*

(b) *the graph of t is closed for $\xi \times \sigma(F, H)$.*

Then t is continuous under ξ and the norm topology $\beta(F, H)$.

(iii) *The same assertion as* (ii) *with F and H replaced by $l_\infty(A)$ and $l_1(A)$, respectively, where $|A| = a$.*

Proof. (i) $\Rightarrow$ (ii). Let $t: E \to F$ be as in (ii) and consider its transpose $t': F' \to E^*$ (where $F' = H''$). By hypothesis, $t'^{-1}(E')$ contains a $\sigma(H, F)$-dense subspace of H. Since such a subspace is also dense in the Banach space H, each element y' of H is the limit (under $\tau(H, F)$) of a sequence (y'_n) in $t'^{-1}(E')\cap H$. Since E is also σ-barrelled, $\{t'(y'_n) : n \in N\}$ is an equicontinuous subset of E', and so its $\sigma(E^*, E)$-closure is contained in E'. This implies that $t'(y') \in E'$ and, consequently, $H \subseteq t'^{-1}(E')$.

The closed unit ball B of H is dense in the closed unit ball B' of F' under $\sigma(F', F)$. If D is any dense subset of the Banach space H with $|D| \leqslant a$, then $D \cap B$ is a $\sigma(F', F)$-dense subset of B' of cardinality at most α. Since $t'(D \cap B)$ is $\sigma(E', E)$-bounded, it is equicontinuous by hypothesis. It now follows as before that $t'(B')$ is an equicontinuous subset of E', so that t is continuous as required.

(ii) $\Rightarrow$ (iii). It is easily seen that $l_1(A)$ has density character $|A|$ for any infinite set A. Consequently, (iii) is a special case of (ii).

(iii) $\Rightarrow$ (i). Let C be any non-empty $\sigma(E', E)$-bounded set with $|C| \leqslant \alpha$. By introducing a repeated term if necessary, we may write the elements of C as a family $\{x'_\lambda\}_{\lambda \in A}$, where $|A| = \alpha$. Consider the mapping $t\colon E \to l_\infty(A)$ defined by

$$t(x) = (\langle x, x'_\lambda \rangle)_{\lambda \in A}.$$

The $\sigma\big(l_1(A),\ l_\infty(A)\big)$-dense subspace

$$M = \big\{(\xi_\lambda)_{\lambda \in A} \colon |\{\lambda\colon \xi_\lambda \neq 0\}| < \aleph_0\big\}$$

of $l_1(A)$ is mapped by the transpose t' of t into E', for if $x \in E$ and $(\xi_\lambda)_{\lambda \in A} \in M$, then

$$\langle x, t'\big((\xi_\lambda)\big)\rangle = \big\langle(\langle x, x'_\lambda\rangle), (\xi_\lambda)\big\rangle = \sum \xi_\lambda \langle x, x'_\lambda\rangle = \Big\langle x, \sum \xi_\lambda x'_\lambda\Big\rangle.$$

This shows that the graph of t is closed under $\xi \times \sigma\big(l_\infty(A),\ l_1(A)\big)$, and so, by hypothesis, t is continuous under ξ and the usual norm topology of $l_\infty(A)$. We note finally that if B is the closed unit ball of $l_\infty(A)'$, then

$$C = t'\big(\{(\delta_{\lambda\mu})_{\mu \in A} \colon \lambda \in A\}\big) \subseteq t'(B),$$

which is equicontinuous.

Remarks. 1. If $\alpha = \aleph_0$, then we are concerned in (ii) with the duals of separable Banach spaces, and the spaces in (iii) are the ordinary l_1 and l_∞. In the case $\alpha = \mathfrak{c}$, the spaces in (ii) are the duals of Banach spaces with linear dimension at most $\mathfrak{c}$.

2. A simple modification of the argument in [9] for ω-barrelled spaces shows that α-barrelled spaces have the countable codimensional subspace property.

PROBLEM (**P 1051**). Let X be a completely regular space and let $C_c(X)$ be the space of real-valued continuous functions on X with the topology of compact convergence. When is $C_c(X)$ a $G(\alpha)$-barrelled space?

We note that in [1], Théorème 4.1, (a) $\Leftrightarrow$ (e), Buchwalter and Schmets give a characterization of σ-barrelledness for $C_c(X)$, which easily extends to describe α-barrelledness for such spaces. Also, in [14], Theorem 9, it is shown that σ-barrelledness and $G(\aleph_0)$-barrelledness are equivalent for $C_c(X)$.

Added in proof. Two related articles by A. Marquina (*A note on the closed graph theorem*, Archiv der Mathematik (Basel) 28 (1977), p. 82-85) and H. Pfister (*Über das Gewicht und den Überdeckungstyp von uniformen Räumen und einige Formen des Satzes von Banach-Steinhaus*, Manuscripta Mathematica 20 (1977), p. 51-72) have appeared after submission of this paper.

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Reçu par la Rédaction le 19. 11. 1976