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ON HARMONIC SEPARATION

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0. Let A and B be disjoint subsets of a locally compact Abelian group Γ, and let G be the dual of Γ. The following condition was introduced in [8] and discussed further in [3]:

(M) There is a constant $C > 0$ such that for any pair of trigonometric polynomials

$$P^{(1)}(\cdot) = \sum_{\text{finite}} a_n^{(1)}(\gamma_n^{(1)}, \cdot) \quad \text{and} \quad P^{(2)}(\cdot) = \sum_{\text{finite}} a_n^{(2)}(\gamma_n^{(2)}, \cdot),$$

where $\gamma_n^{(1)} \in A$, $\gamma_n^{(2)} \in B$, we have

$$\|P^{(1)}+P^{(2)}\|_\infty \geqslant C(\|P^{(1)}\|_\infty + \|P^{(2)}\|_\infty).$$

If A and B satisfy (M), we call them *harmonically separated* (h.s.). It is obvious that (M) is equivalent to

$$(\mathrm{M}') \qquad \|\mu_1+\mu_2\|_{PM} \geqslant C(\|\mu_1\|_{PM} + \|\mu_2\|_{PM})$$

for any pair of discrete measures $\mu_1 \in M(A)$ and $\mu_2 \in M(B)$, where $\|\mu\|_{PM} = \|\hat{\mu}\|_\infty$ ($\hat{\mu}$ — the Fourier transform). It is also clear that the inequality in (M) extends to all pairs of almost periodic functions on G whose Fourier exponents belong to A and B, respectively. This in turn means that the Fourier series $\sum$ of any almost periodic function the exponents of which belong to $A \cup B$ splits into two parts $\sum_1$ and $\sum_2$, Fourier series of almost periodic functions with exponents in A and B, respectively.

By duality we obtain one equivalent condition more. For any compact group G let $C_E(G)$ denote the space of continuous functions with spectrum in E. Let $\tilde{G}$ denote the Bohr compactification of G. We saw that (M) means that the following decomposition holds:

$$(1) \qquad C_{A\cup B}(\tilde{G}) = C_A(\tilde{G}) + C_B(\tilde{G}).$$

The corresponding decomposition holds for the dual Banach space $C^*_{A\cup B}(\tilde{G})$. Thus the Dirac measure δ_0 on G is representable as $\nu_1+\nu_2$, where $\hat{\nu}_1|A \equiv 0$, $\hat{\nu}_1|B \equiv 1$. Conversely, if such ν_1 and ν_2 exist, then — setting $f = f*\nu_1 + f*\nu_2$ for any $f \in C_{A\cup B}(\tilde{G})$ — we get (1). Hence we have

PROPOSITION 1. (M) *holds if and only if there exists a measure ν on G such that* $\hat{\nu}\,|A \equiv 0$, $\hat{\nu}\,|B \equiv 1$. (ν is called a *separating measure.*)

By this proposition and by (1) we obtain

PROPOSITION 2. (M) *implies that, for* $1 \leqslant p \leqslant \infty$,

$$L^p_{A\cup B}(\tilde{G}) = L^p_A(\tilde{G})\oplus L^p_B(\tilde{G})$$

and (M) *is equivalent to this decomposition for* $p = \infty$.

To prove the first part one may use the convolution with ν while the second follows from the equivalence of norms in C and L^∞. We note that the space $L^p(\tilde{G})$ can be regarded as the space of B^p-almost periodic functions (i.e. in the sense of Besicovitch) on G.

Two sets A and B will be called *fully harmonically separated* (fully h.s.) if they can be enlarged to some sets A' and B' such that A' and B' are still h.s. and $A'\cup B' = \Gamma$. By Proposition 1 we see at once that this condition is equivalent to the existence of an idempotent measure whose Fourier transform equals 1 exactly on A'. Hence all pairs of sets which are fully h.s. can be described by means of known Cohen's theorem, whereas the problem of decomposition of a set E into two h.s. parts reduces to the study of "relative idempotents" on E, i.e. idempotents of $B(\Gamma)/J_E$, where $B(\Gamma)$ is the algebra of Fourier-Stieltjes transforms and J_E denotes the corresponding ideal in it. No simple characterization of such idempotents seems to be known or even possible. Examples in $\boldsymbol{Z}$ or $\boldsymbol{R}$ we intend to give are aimed to shed some light on this problem.

1. Owing to the theorem of Cohen it is very easy to find examples of sets of integers which are h.s. in $\boldsymbol{Z}$ (or in other groups). The set of even and that of odd integers may be the simplest ones, besides the trivial case A finite, B arbitrary. Actually, it is the easier part of Cohen's theorem (a sufficient condition for a set to be the support of an idempotent in $B(\Gamma)$) that is needed in examples of this kind. The difficult part (necessary conditions) allows us to find pairs of not h.s. complementary sets such as $\boldsymbol{Z}^+$ and $\boldsymbol{Z}^-$, a well-known negative instance. In general, it seems to be more difficult to prove that some given sets are not h.s. that to state the contrary if it occurs. Here are two problems concerning not full harmonic separation:

1° find sets E in $\boldsymbol{Z}$ (if there are any) such that $E = A\cup B$ and $A\cap B = \emptyset$ imply that A and B are h.s. (extreme harmonic decomposibility);

2° find sets E in $\boldsymbol{Z}$ (if there are any) such that $E = A\cup B$, A and B being infinite, implies that A and B are not h.s. (harmonic indecomposibility).

The answer to the first problem is immediate: they are precisely Sidon sets (see, e.g., [10], p. 4). The answer to the second problem (raised by M. Bożejko) was recently found by C. McGehee: h.s. sets do not exist (not published yet). We return to this subject later on for a related problem.

We call a sequence (γ_k) *uniformly distributed* (u.d.) in an l.c.a. group if

$$\frac{1}{k}[f(\gamma_1)+\ldots+f(\gamma_k)]\to\mathfrak{M}(f) \text{ (mean value)}$$

holds for any almost periodic function on Γ (cf. [9], p. 295).

We recall the known theorem of H. Weyl stating that for any increasing sequence of integers (n_k) the sequence $(n_k a)$ is u.d. (mod 1) for almost every real number a. The sequence (n_k) may be called of *first* (*second*) *kind* if the set of a's for which $(n_k a)$ is not u.d. (mod 1) is countable (uncountable). Finally, we say that $E\subset \mathbf{Z}$ is a *Kaufman set* (Ka-set) if there exists a continuous measure on $\mathbf{T}$ such that

$$\inf_E |\hat{\mu}(n)| > 0.$$

The (*relative*) *density* of a set $\Lambda\subset\mathbf{Z}$ with respect to an increasing sequence $E = (a_n)_{n=-\infty}^{\infty}$ of integers will mean

$$\lim_{N\to\infty}\frac{1}{2N}\left|\{|n|<N:\ a_n\in\Lambda\}\right|.$$

Replacing lim by $\overline{\lim}$ ($\underline{\lim}$) we get the upper (lower) relative density. The modification for sets and sequences in $\mathbf{Z}^+$ is obvious. If $\Lambda\subset E$, we say also "the density of Λ in E". We occasionally do not distinguish between a set of integers and the increasing sequence of its elements.

LEMMA 1. *A Ka-set has density* 0 *with respect to every sequence of first kind.*

Proof. We have but to reproduce with some caution the argument used in [6], Theorem 12. Let us suppose to the contrary that some sequence (n_k) of first kind contains a Ka-set Λ such that for a sequence of indices $k_1<k_2<\ldots$ we have

$$\forall_s \frac{1}{k_s}|\Lambda\cap\{n_1,\ldots,n_{k_s}\}|>\varepsilon>0. \tag{2}$$

Let μ be a continuous measure such that

$$\inf_\Lambda|\hat{\mu}(n)|>0.$$

We put $\nu=\mu*\mu^*$; then $\hat{\nu}=|\hat{\mu}|^2$, $\inf_\Lambda\hat{\nu}(n)>\delta$ for some $\delta>0$, and ν is continuous. Since (n_k) is of first kind, ν vanishes identically on the set of α's for which $(n_k \alpha)$ is not u.d. (mod 1). Thus, on account of Weyl's criterion, we have

$$\lim_K \frac{1}{K} \sum_{k=1}^{K} \exp(-2\pi i n_k \alpha) = 0 \quad \nu\text{-a.e.}$$

Since

$$\int_T \exp(-2\pi i n_k \alpha) d\nu(\alpha) = \hat{\nu}(n_k),$$

by the Lebesgue bounded convergence theorem we get

$$\lim_K \frac{1}{K} \sum_{k=1}^{K} \hat{\nu}(n_k) = 0.$$

On the other hand, $\hat{\nu}(n) > \delta$ for each $n \in \Lambda$ and $\hat{\nu}(n) \geqslant 0$ for any n, whence, by (2),

$$\lim_s \frac{1}{k_s} \sum_{k=1}^{k_s} \hat{\nu}(n) \geqslant \varepsilon\delta,$$

a contradiction.

Let us call a point $p \in \tilde{Z}$ a *density point* of $E \subset Z$ if every neighbourhood of p in $\tilde{Z}$ intersects E in a set whose upper density with respect to E (naturally ordered) is positive.

THEOREM 1. *If E_1 and E_2 are sets of first kind in Z having a common density point, then they are not h.s.*

Proof. Suppose the contrary. Then there exists a measure $\mu \in M(T)$ such that $\hat{\mu}(n) = 0$ if $n \in E_1$ and $\hat{\mu}(n) = 1$ if $n \in E_2$. Let $\mu = \mu_c + \mu_d$, where μ_c is continuous and μ_d is discrete. For $\varepsilon > 0$ we have $|\hat{\mu}_c(n)| < \varepsilon$ outside a Ka-set Λ. By Lemma 1 the set $\Lambda \cap E_i$ has density 0 in E_i $(i = 1, 2)$. Hence the assumption implies that the sets $E_1 \setminus \Lambda$ and $E_2 \setminus \Lambda$ have still a common cluster point in $\tilde{Z}$. But we have $|\hat{\mu}_d(n)| < \varepsilon$ for $n \in E_1 \setminus \Lambda$ and $|\hat{\mu}_d(n)| > 1 - \varepsilon$ for $n \in E_2 \setminus \Lambda$. Since $\hat{\mu}_d$ is an almost periodic function (i.e. continuously extendable on $\tilde{Z}$), this leads to a contradiction if we choose $\varepsilon < \frac{1}{2}$.

LEMMA 2. *If an increasing sequence (n_k) is u.d. in Z, then (n_k) is of first kind.*

Proof. If t is real not integer, then $\mathfrak{M}(e^{2\pi i t \cdot}) = 0$, whence for a u.d. sequence (n_k) we have

$$\lim_k \frac{1}{k} \sum_{j=1}^{k} \exp(2\pi i t n_j) = 0.$$

If t is irrational, then the same holds for lt instead of t, where l is any integer not equal to 0. Now from Weyl's criterion it follows that (tn_k) is u.d. (mod 1).

THEOREM 2. *If $E_1 \subset Z$ is u.d. and $E_2 \subset Z$ is of first kind, then E_1 and E_2 are not h.s.*

Proof. Since E_2 is infinite, it has a density point in view of the compactness of $\tilde{Z}$. But every point of $\tilde{Z}$ is a density point of E_1. Hence it is enough to apply Lemma 2 and Theorem 1.

LEMMA 3. *If $s \in Z^+$, then 0 is a density point of $(n^s)_{n=1}^{\infty}$.*

Proof. Let U be a neighbourhood of 0 in $\tilde{Z}$ determined by some $\varepsilon > 0$ and some real numbers $a_1, \dots, a_r$. So $n \in U$ iff

$$|\exp(2\pi i n a_j) - 1| < \varepsilon \qquad (j \in [1, r]).$$

Let $\beta_0 = 1, \beta_1, \dots, \beta_\nu$ be any system independent over Q and such that its linear span over Q contains all a_j. There exist a $Q \in Z^+$ and integers $p_l^{(j)}$ such that

$$a_j = \sum_{l=1}^{\nu} \frac{p_l^{(j)}}{Q} \beta_l \qquad (j \in [1, r]).$$

Thus

$$Q^s a_j \equiv \sum_{l=1}^{\nu} Q^{s-1} p_l^{(j)} \beta_l \pmod 1 \qquad (j \in [1, r]). \tag{3}$$

If $h_1, \dots, h_\nu$ are integers not all 0, then

$$\eta = Q^{s-1} \sum_{l=1}^{\nu} h_l \beta_l$$

is irrational, and so the sequence $(n^s \eta)$ is u.d. (mod 1) (a well-known theorem of Weyl). Consequently ([1], p. 71), the ν-fold sequence $(Q^{s-1}\beta_l n^s)_{n=1}^{\infty}$ $(1 \leqslant l \leqslant \nu)$ is u.d. (mod 1) in the ν-dimensional torus. Thus, for every $\delta > 0$, the set of n's such that

$$|\exp(2\pi i Q^{s-1} \beta_l n^s) - 1| < \delta \qquad (l \in [1, \nu]) \tag{4}$$

has positive density. In view of (3), δ can be chosen in a way such that (4) implies

$$|\exp(2\pi i Q^s a_j n^s) - 1| < \varepsilon \qquad (j \in [1, r]);$$

in other words, $Q^s n^s \in U$. Obviously, the set of numbers Qn which enter into these inequalities has still a positive density in Z^+, which completes the proof.

THEOREM 3. *If $\alpha, \beta > 0$, then the sets* $([n^\alpha])$ *and* $(-[n^\beta])$ *are not h.s.* ($[x]$ *denotes the integer part of* x).

Proof. If α is not an integer, then the sequence (n^α) is u.d. in $\boldsymbol{R}$ [2]. Therefrom it follows [4] that $([n^\alpha])$ is u.d. in $\boldsymbol{Z}$, and so is $(-[n^\alpha])$. If α is integer, then (n^α) is of first kind (Weyl). Consequently, if α or β is not an integer, then the assertion follows immediately from Lemma 2 and Theorem 2. If α and β are both integer, we apply Lemma 3 and Theorem 1 and we are done.

THEOREM 4. *The set P of primes and the set $-P$ are not h.s.*

Proof. We are going to prove that 1 and -1 are density points of P. So they are also density points of $-P$. Now, the known theorem of Vinogradov asserts that the sequence $(p\alpha)_{p\in P}$ is u.d. (mod 1) for every irrational α. Hence P and $-P$ are of first kind and Theorem 1 can be used to achieve the proof.

It will be sufficient to show that 1 is a density point of P. The proof for -1 is quite analogous. So let U denote an arbitrary neighbourhood of 1 in $\boldsymbol{Z}$, determined by an $\varepsilon > 0$ and some numbers $a_1, \ldots, a_r$. Thus

$$U \cap \boldsymbol{Z} = \{n \in \boldsymbol{Z}\colon\ |\exp(2\pi i a_j n) - \exp(2\pi i a_j)| < \varepsilon\} \qquad (j \in [1, r]).$$

Let $\beta_0 = 1, \beta_1, \ldots, \beta_\nu$ be numbers independent over $\boldsymbol{Q}$ and such that each a_j is a linear combination of β's with rational coefficients. Let Q denote a common denominator of these coefficients. We have $p \in U$ if and only if all inequalities

$$\big|\exp\big(2\pi i a_j(p-1)\big) - 1\big| < \varepsilon$$

are satisfied. So if $\delta > 0$ is properly chosen, then $p \in U$ is implied by inequalities

$$\left|\frac{\beta_l}{Q}p - \frac{\beta_l}{Q}\right| < \delta \ (\mathrm{mod}\ 1) \qquad (l \in [0, \nu])$$

and, the more, by the following system of relations in which P_Q means $P \cap (1 + mQ)_{m=1}^{\infty}$:

$$(5) \qquad p \in P_Q, \qquad \left|\frac{\beta_l}{Q}p - \frac{\beta_l}{Q}\right| < \delta \ (\mathrm{mod}\ 1) \qquad (l \in [1, \nu]).$$

The crucial point comes now: not only $(p\alpha)_{p\in P}$ is u.d. (mod 1) for every irrational but the same holds for the sequence of primes in any arithmetical progression in which more than one prime occurs. The author is indebted to Prof. Schinzel for having proved it (by adapting the original proof of Vinogradov) and for his permission to use it here.

Since $\sum_{l=1}^{\nu} h_l\beta_l = 0$ and $h_l \in Z$ imply $h_l = 0$ $(l \in [1, \nu])$ it follows from the statement above that the ν-fold sequence $(p\beta_l/Q)$ $(p \in P_Q)$ is u.d. (mod 1) in ν-dimensional torus. Thus the values of p satisfying the inequalities in (5) form a sequence of positive density in P_Q. By Dirichlet's theorem the density of P_Q in P is $1/Q$, whence (5) is completely satisfied by primes belonging to a set of positive density in P. So the proof of Theorem 4 is completed.

2. We would like to give some more examples of harmonic separation.

PROPOSITION 3. *If U_1 and U_2 are open sets in $\tilde{\Gamma}$ with disjoint closures, then $\Gamma \cap U_1$ and $\Gamma \cap U_2$ are h.s.*

Proof. Since the algebra $A(\Gamma)$ is regular, it contains a function equal to 0 on U_1 and to 1 on U_2. Hence, as $G_d = (\tilde{\Gamma})^{\wedge}$, there is a discrete measure μ on G such that $\hat{\mu} = 0$ on $U_1 \cap \Gamma$ and $\hat{\mu} = 1$ on $U_2 \cap \Gamma$. Now Proposition 1 completes the proof.

H.s. sets in Z constructed in that way are relatively dense in Z, and so they are never Sidon sets. In view of Lemma 1 and Proposition 3 we point out that the role of continuous measures in harmonic separation is by no means negligible; for example, the sets $A = (n!)_1^\infty$ and $B = (n!+n)_1^\infty$ are h.s., since they are both Sidon. However, there exists no discrete measure the Fourier transform of which would be equal to 0 on A and to 1 on B, since there is no almost periodic function on Z having this property, as it is easily seen by means of "almost periods".

Sequences $(n!)$ and $(-n!)$ furnish an example of h.s. but not fully h.s. sets. In fact, they are h.s., since they are Sidon sets. On the other hand, let us suppose that they are fully h.s. Hence there exists a decomposition $Z = A \cup B$ into two h.s. sets such that $(n!) \subset A$ and $(-n!) \subset B$. The set A, by assumption and by Proposition 1, is the support of an idempotent measure. By Cohen's theorem the support S of $\hat{\mu}$ for μ idempotent differs only for a finite set from a finite union of full (two-sided infinite) arithmetical progressions. Let us refer to those progressions as entering into S. Since, for an arbitrary $m \in Z^+$, all but a finite number of terms $n!$ are divisible by m, among full progressions entering into A there is at least one that contains 0. Thus there exists an m for which all elements of the progression $(km)_{k=-\infty}^{\infty}$ except a finite number belong to A, which is impossible since the negative terms belong to B.

Actually, the argument above shows that the sequence $(-n!)_1^\infty \cup (n!)_1^\infty$ has the property of not containing any two infinite subsets which belong to disjoint arithmetical progressions. This property is obviously hereditary. Let us denote it by (P_1). We have

PROPOSITION 4. *Property (P_1) is equivalent to each of the following conditions:*

(1) *The sequence is convergent in polyadic metric (i.e. in p-adic metric for each prime to the same limit).*

(2) *The set cannot be decomposed into two infinite fully h.s. parts.*

For the proof one needs, besides the theorem of Cohen, only some elementary arithmetics, close to that used in the special case of $(-n!)\cup(n!)$.

To obtain a pair of h.s. sets which are not fully h.s. and non-Sidon, one can take any two cluster points of $E=(n!)_1^\infty$ in $\tilde{Z}$ and any two compact neighbourhoods separating them. Then $E_1=U_1\cap Z$ and $E_2=U_2\cap Z$ are h.s. by Proposition 3, not fully h.s. since $U_1\cap E$ and $U_2\cap E$ are not, and non-Sidon because they are relatively dense in Z. An explicit example of a set having the desired properties will be given in the next section.

3. Let us return to the problem of Bożejko formulated in Section 0 and solved already in the negative: does there exist a harmonically indecomposable set E of integers? This problem is closely related to the following one, which the author raised some years ago [5]: does there exist a set E of integers such that there exists no almost periodic function (on Z) taking values 0 or 1 for every $n\in E$, each of them infinitely many times? (1) Obviously, a harmonically indecomposable set must have this property. Let us call it (P_2).

PROPOSITION 5. $(P_2)\Rightarrow(P_1)$.

To see this let us observe that the group $\tilde{Z}$ may be written as the product $\prod_p I_p\oplus\tilde{R}$, corresponding to the representation of T_d as $\sum_p C_{p^\infty}+R_d$. It is obvious that if the projection of E on one of the "axes" $\prod_p I_p$ or $\tilde{R}$ decomposes into two infinite topologically separated sets, then the same holds for E itself, whence we infer immediately that (P_2) fails. Since $\prod_p I_p$ is totally disconnected, (P_2) cannot hold unless the projection of E on the first axis has only one accumulation point, but this simply means that E converges in polyadic metric. Proposition 4 completes the proof.

The sequence $(n!)$ is the simplest one fulfilling condition (P_1), but it has a property just opposite to (P_2): every bounded function on $(n!)$ is extendable to an almost periodic function [13]. Thus we are led to modify $(n!)$ in such a way that its Sidonicity (and, consequently, the interpolation property above) be destroyed but the convergence to 0 in each p-adic metric be preserved. The simplest way is to take $1, 2, 4, 6, 12, 18, 24, 48, \ldots$, i.e. the sequence C built of blocks $\{n!, 2n!, \ldots, nn!\}$. We are not able to decide whether C has property (P_2). Let us observe that the latter fails if we replace C by C_γ by shortening the blocks in C to $\{n!, 2n!, \ldots, [n\gamma]n!\}$, where $\gamma\in(0,1)$. In fact, let us select from every such block exactly the terms $kn!$ with k even. Then they form together

(1) Recently answered in the affirmative by G. S. Woodward (to appear in this journal).

a set A which is h.s. from $C_\gamma \setminus A = B$. To prove this assertion we can verify, by an easy computation, that the sets

$$\left\{n_i \frac{e}{2} (\mathrm{mod}\, 1)\right\} \; (n_i \in A) \quad \text{and} \quad \left\{m_i \frac{e}{2} (\mathrm{mod}\, 1)\right\} \; (m_i \in B)$$

have disjoint closures in $\boldsymbol{T}$. Thus A and B have disjoint closures as subsets of $\tilde{\boldsymbol{Z}}$. Since, in view of Proposition 4, C_γ does not decompose into two fully h.s. sets and since it is not a Sidon set, the pair (A, B) fulfils the announcement at the end of Section 2.

4. We go over to problems of harmonic separation on the real line. Again, as in $\boldsymbol{Z}$, there is no difficulty in producing examples of full harmonic separation of two sets. Actually, that is the problem of decomposing $\boldsymbol{R}$ into two h.s. sets or, by Proposition 1, the problem of idempotents in the algebra $B(\boldsymbol{R}_d)$. Thus Cohen's theorem yields a general solution. The set of rationals and its complement are h.s., for the set of integers and its complement the same is true, etc. It is obvious from the definition that two sets of integers which are h.s. in $\boldsymbol{Z}$ are h.s. in $\boldsymbol{R}$ and conversely. It follows from Cohen's theorem that if two sets in $\boldsymbol{Z}$ are fully h.s. in $\boldsymbol{R}$, then they are fully h.s. in $\boldsymbol{Z}$ and conversely.

To show h.s. not fully h.s. dense sets in $\boldsymbol{R}$ we take two reals a_1, a_2 independent over $\boldsymbol{Q}$, and two disjoint intervals I_1 and I_2. Let H be the group of all numbers $x = t_1 a_1 + t_2 a_2$, where $t_1 = t_1(x)$ and $t_2 = t_2(x)$ are of the form $r/2^l$ ($r \in \boldsymbol{Z}$, $l \in \boldsymbol{Z}^+$), i.e. they are dyadic rationals. Let A be the set of all $x \in H$ such that $t_1(x) \in I_1$. Let B be defined analogously with I_2 instead of I_1.

THEOREM 5. *The sets A and B, as defined above, are dense in $\boldsymbol{R}$, h.s., not fully h.s., and non-Sidon in $\boldsymbol{R}_d$.*

Proof. For any $\tau > 0$, $\exp(i\tau t_1(\cdot))$ is a character of H. We can choose τ so small that $E_1 = \{e^{i\tau t}: t \in I_1\}$ and $E_2 = \{e^{i\tau t}: t \in I_2\}$ be separated subsets of $\boldsymbol{T}$. Let φ be any character of $\boldsymbol{R}_d$ such that $\varphi | H = \exp(i\tau t_1(\cdot))$. Then the sets $H \cap \varphi^{-1}(E_1) = A$ and $H \cap \varphi^{-1}(E_2) = B$ are separated with respect to the topology which the Bohr compactification $(\boldsymbol{R}_d)^{\sim}$ induces in $\boldsymbol{R}$. Hence they are h.s. by Proposition 3. It is obvious that they are dense in $\boldsymbol{R}$.

Let us suppose to the contrary that they are fully h.s. So there exist sets $A_1 \supset A$ and $B_1 \supset B$ and a function $f \in B(\boldsymbol{R}_d)$ such that $A_1 \cup B_1 = \boldsymbol{R}$, $f | A_1 = 0$, $f | B_1 = 1$. Let

$$H_1 = \{x \in H: t_2(x) = 0\}.$$

The function $f | H_1$ is an idempotent of $B(H_1)$, and so $H_1 \cap A_1$ and $H_1 \cap B_1$ are members of the coset ring of H_1. Any such a member is of the form

$$(6) \qquad \bigcup_{1=i}^{s} A^{(i)} \cap B_1^{(i)} \cap \ldots \cap B_{r_i}^{(i)},$$

where $A^{(i)}$ are cosets and $B_j^{(i)}$ are (set-theoretical) non-void complements of cosets. Since $A_1 \supset A$ and $B_1 \supset B$, both sets $H_1 \cap A_1$ and $H_1 \cap B_1$ contain intervals in H_1. Since H_1 is isomorphic to the group of dyadic rationals, all proper subgroups of H_1 are cyclic, and so non-dense in $\boldsymbol{R}$. Hence all proper cosets are non-dense. Therefore, among the $A^{(i)}$ occurring in representation (6) of $H_1 \cap A_1$ ($H_1 \cap B_1$) at least one is the whole group H_1. The intersections $\bigcap_j B_j^{(i)}$ are complements of unions of proper cosets of H_1, so they are dense in H_1. Then both $H_1 \cap A_1$ and $H_1 \cap B_1$ are dense in H_1. This means a contradiction which proves that A and B are not fully h.s. They are not Sidon in $\boldsymbol{R}_d$. In fact, otherwise they would be Sidon in H, which is impossible since each of them contains an infinite coset in H, e.g. $\{x \in H: t_1(x) = c\}$ for $c \in I_1$ or $c \in I_2$, respectively.

The essential point in the argument above is that two sets in the group H_1, each of which contains an interval in H_1, are never fully h.s. The same can be proved for the whole group of rationals (I owe to H. Rindler a skillful technique for doing it) and used in an analogous manner to produce examples of "bigger" sets still satisfying the assertion of Theorem 5. Namely, we fix a Hamel basis $t_1, t_2, \ldots, t_a, \ldots$ in $\boldsymbol{R}$, and assume that A (B) is the set of all $x \in \boldsymbol{R}$ such that, in the representation of x with respect to (t_a), the (rational) t_1-coordinate belongs to I_1 (I_2), where I_1 and I_2 are disjoint segments.

The h.s. property of a pair of sequences in $\boldsymbol{R}$ can sometimes be stated if one knows that some other specifically related sequences are h.s. We now prove a theorem in this direction.

THEOREM 6. *Let (τ_n) be a strictly monotone sequence of positive numbers tending to infinity or to zero and such that $\tau_i\tau_j \in (\tau_n)$ for every i, j. Then if the sets $(-\tau_n)$ and (τ_n) are h.s., so are the sets $(-\tau_n^{-1})$ and (τ_n^{-1}).*

Proof. According to a theorem of Głowacki [3] a countable set $A = (t_i)_{i=1}^{\infty} \subset \boldsymbol{R}^+$ and a closed set $B \subset \boldsymbol{R}^- \cup \{0\}$ are h.s. if and only if there exists a sequence (f_n) of functions in $A(\boldsymbol{R})$ such that $f_n(t_i) = 1$ for $i \in [1, n]$, $f_n(t) = 0$ for $t \in B$ and the norms $\|f_n\|_A$ are bounded. So let (f_n) be such a sequence chosen for $A = (\tau_i)$ and $B = (-\tau_i) \cup \{0\}$. We put

$$\pi_n = \prod_{i=1}^{n} \tau_i.$$

Then π_n occurs in the sequence (τ_n) with some index $\operatorname{ind}\pi_n$. We fix numbers $m_n \geqslant \operatorname{ind}\pi_n$ and put $\tilde{f}_n(t) = f_{m_n}(t\pi_n)$ for any n. Then, if $i \leqslant n$, we have

$$\tilde{f}_n(\tau_i^{-1}) = f_{m_n}(\tau_i^{-1}\pi_n) = f_{m_n}(\tau_1 \ldots \tau_{i-1}\tau_{i+1} \ldots \tau_n) = 1$$

and, for any i,

$$\tilde{f}_n(-\tau_i^{-1}) = \tilde{f}_n(0) = 0.$$

On the other hand, the functions $\tilde{f}_n$ and f_{m_n} have the same A-norm, so the norms $\|\tilde{f}_n\|_A$ are bounded. Thus, by Głowacki's theorem used in the opposite direction, the sequences (τ_n^{-1}) and $(-\tau_n^{-1})$ are h.s.

COROLLARY. *For any rational a the sequences (n^a) and $(-n^a)$ are not h.s.*

For the proof it is enough to use Theorem 3.

It seems that for irrational values of a the sequences (n^a) and $(-n^a)$ can be h.s. If this assertion holds for $a > 0$, then it holds for $a < 0$ as well, and conversely — owing to Theorem 6. We are not able to prove or to disprove this conjecture. (**P 1135**)

5. Let us say a few words about harmonic separation in the torus. Let $\boldsymbol{T}$ be the 1-dimensional torus with characters written as $e^{2\pi itn}$ $(-1/2 \leqslant t < 1/2, n \in \boldsymbol{Z})$. We identify it with the interval $[-1/2 \leqslant t < 1/2)$.

THEOREM 7. *If any sets $A, B \subset [-1/2, 1/2)$ are h.s. as subsets of $\boldsymbol{T}$, then they are h.s. as subsets of $\boldsymbol{R}$. The converse fails.*

Proof. By Proposition 1 there exists a measure $\mu \in M(\tilde{\boldsymbol{Z}})$ such that $\hat{\mu} = 0$ on A and $\hat{\mu} = 1$ on B. We can consider $\tilde{\boldsymbol{Z}}$ as a (closed) subgroup of $\tilde{\boldsymbol{R}}$, whence μ appears as a member of $M(\tilde{\boldsymbol{R}})$ with support in $\tilde{\boldsymbol{Z}}$. Every character of $\tilde{\boldsymbol{Z}}$ is the (unique) extension of $e^{2\pi it\cdot}$ for some $t \in \boldsymbol{T}$ and it is extendable (in many ways) to a character of $\tilde{\boldsymbol{R}}$. Hence the values of $\hat{\mu}$ on $(\tilde{\boldsymbol{Z}})\hat{} = \hat{\boldsymbol{Z}}_d = \boldsymbol{T}_d$ are the same whether we regard μ as a measure on $\tilde{\boldsymbol{Z}}$ or on $\tilde{\boldsymbol{R}}$. This proves the first part of the theorem.

We now set $A = [-1/2, 0)$ and $B = [1/4, 1/2)$. The sets A and B are of course h.s. in $\boldsymbol{R}$. We show that they are not h.s. in $\boldsymbol{T}$. It is immediate that the h.s. property in any l.c.a. group is invariant under the shift $x \to x + a$. Hence, if A and B were h.s. in $\boldsymbol{T}$, so it would be the same for $A + 1/2$ and $B + 1/2$ (mod 1), that is for the sets $[0, 1/2)$ and $[-1/4, 0)$. Then they would be h.s. in $\boldsymbol{R}$ but it is not the case as is well known (and follows at once from the Corollary to Theorem 6).

THEOREM 8. *Let T be the torsion part of $\boldsymbol{T}$. If $A, B \subset T$ are h.s. in $\boldsymbol{R}$ and if $\mathrm{Gp}(A \cup B) \cap T = \{0\}$, then A and B are h.s. in $\boldsymbol{T}$.*

In fact, more can be proved: if

$$P(x) = \sum_{k=1}^{n} a_k \exp(2\pi i t_k x) \quad (t_k \in A \cup B),$$

then

$$\sup_{x \in \boldsymbol{R}} |P(x)| = \sup_{n \in \boldsymbol{Z}} |P(n)|. \tag{7}$$

From the assumption it follows that there is a set $\{\beta_1, \ldots, \beta_m\} \subset \boldsymbol{R}$ such that the numbers $1, \beta_1, \ldots, \beta_m$ are independent over $\boldsymbol{Q}$ and, for every $k \in [1, n]$,

$$t_k = \sum_{j=1}^{m} \frac{p_j^{(k)}}{Q} \beta_j,$$

where $p_j^{(k)}$ and Q are integers. Substituting these expressions in P and using Kronecker's theorem in a standard way, we can find, for every $x \in \boldsymbol{R}$, a suitable n for which $P(n)$ comes arbitrarily near to $P(x)$. This obviously implies (7).

6. The h.s. property plays a role in some problems concerning unions of special sets. We recall that a set Λ in a discrete Abelian group Γ is called a *Rosenthal set* (Ro-set) if any function $f \in L^\infty_\Lambda(G)$ is continuous. In non-discrete l.c.a. groups this notion splits into two, Ro and Ro*. The first one is defined just as above and the latter means that $f \in L^\infty_\Lambda(G)$ implies $f \in \mathrm{AP}(G)$ (i.e. the Bohr almost periodicity) [7]. A set Λ in a discrete group Γ is called a *set of uniform convergence* (UC-set) if the Fourier series of any function in $C_\Lambda(G)$ is uniformly convergent [12].

THEOREM 9. (a) *Let Λ_1 be an isolated set in an l.c.a. group Γ and let Λ_2 be a closed set in Γ containing all accumulation points of Λ_1. If $\overline{\Lambda_1}$ and Λ_2 are Ro-sets (Ro*-sets) and if Λ_1 and Λ_2 are h.s., then $\Lambda_1 \cup \Lambda_2$ is a Ro-set (Ro*-set).*

(b) *If Λ_1 and Λ_2 are h.s. UC-sets in a discrete Abelian group, then $\Lambda_1 \cup \Lambda_2$ is a UC-set.*

Remark. I am indebted to P. Głowacki for statement (a). My original statement required a stronger assumption. Statement (b) is known.

Proof. (a) Both assertions (for Ro and for Ro*) follow immediately from the representation

$$L^\infty_{\Lambda_1 \cup \Lambda_2} = L^\infty_{\overline{\Lambda_1}} + L^\infty_{\Lambda_2}. \tag{8}$$

Using an approximate identity in $A(\Gamma)$ we see that (8) will be proved as soon as we show that every function in $L^\infty_{\Lambda_1 \cup \Lambda_2}$ whose spectrum is compact is a sum of two functions belonging to $L^\infty_{\overline{\Lambda_1}}$ and $L^\infty_{\Lambda_2}$, respectively. This in turn we obtain by a slight modification of the proof of Theorem 2 (i) in [3]. If the assertion of that theorem is replaced by (8), then the assumption that Λ_2 has spectral synthesis may be dropped.

(b) We decompose the Fourier series $\sum a_n \chi_n$ of any $f \in C_{\Lambda_1 \cup \Lambda_2}(G)$ into two parts according to whether $\chi_n \in \Lambda_1$ or $\chi_n \in \Lambda_2$. Thus we get two Fourier series of continuous functions with spectrum in Λ_1 or in Λ_2, respectively (see Section 0). Since they are both uniformly convergent, so is the series $\sum a_n \chi_n$.

7. In this last section we may introduce some property related to but weaker than harmonic separation. First, let us observe that sets

E_1 and E_2 in Γ are h.s. if and only if, for every sequence of trigonometric polynomials (P_n) whose spectra lie in $E_1 \cup E_2$, the uniform convergence $P_n \rightrightarrows 0$ implies $P_n^{(1)} \rightrightarrows 0$, $P_n^{(1)}$ being the "E_1-part" of P_n according to condition (M) (see Section 0). Actually, the implication

$$(P_n \rightrightarrows 0) \Rightarrow \left(P_n^{(1)}(0) \to 0\right)$$

is an equivalent condition, since

$$P_n^{(1)}(t) \to 0 \quad \text{for all } t \in G$$

is obtained by translation and the uniformity from the boundedness of the functional $P \to P^{(1)}(0)$ for $P \in C_{E_1 \cup E_2}(\tilde{G})$.

If we require only that $P_n \rightrightarrows 0$ implies $P_n^{(1)}(t) \to 0$ for all $t \in G$ provided the A-norms of P_n (i.e. $\|\hat{P}_n\|_1$) are bounded, then we obtain a sufficient (and necessary) condition for the function χ on $E_1 \cup E_2$ taking the value 0 on E_1 and 1 on E_2 to belong to $\overline{B(E_1 \cup E_2)}$ — the uniform closure of $B(E_1 \cup E_2)$. That is a consequence of Theorem 6 in [6] which asserts that a bounded function f on $\Lambda \subset \Gamma$ is in $\overline{B(\Lambda)}$ if and only if (the necessity being very easy), for every sequence of measures $\lambda_n \in M(\Lambda)$, bounded in variation norm, the uniform convergence $\hat{\lambda}_n \rightrightarrows 0$ (on G) implies

$$\int_{\Gamma} f d\lambda_n \to 0.$$

Thus, if we put $\Lambda = E_1 \cup E_2$, $f = \chi$ and if we choose λ_n equal to the measure (with finite support in $E_1 \cup E_2$) whose Fourier transform is $P_n(t + \cdot)$, the statement above appears.

Two sets E_1 and E_2 in Γ are called *weakly harmonically separated* if $P_n \rightrightarrows 0$ and $\|\hat{P}_n\|_1 \leqslant 1$ imply $P_n^{(1)} \rightrightarrows 0$. From the statement just proved it follows that a necessary condition for E_1 and E_2 to be weakly h.s. is that $\chi \in \overline{B(E_1 \cup E_2)}$. A sufficient condition in the case of G compact is obtained if we require moreover that E_1 or E_2 has the Schur property. This means that weak convergence (i.e. pointwise bounded convergence) in C_{E_1} (C_{E_2}) implies strong convergence. This can be the case not only for Sidon sets (Y. Meyer, cf. [11]). On the other hand, Sidon sets are the only sets Λ such that all 0-1 functions on Λ belong to $\overline{B(\Lambda)}$.

The author knows very little about weak harmonic separation. There is essentially but one example of weakly h.s. not h.s. sets he can give at this moment: E_1 an infinite Sidon set and E_2 its complement. In fact, Drury's theorem asserts that in this case $\chi \in \overline{B(E_1 \cup E_2)}$ and the Schur property is implied by Sidonicity.

We do not know whether $\chi \in \overline{B(E_1 \cup E_2)}$ together with the Schur property for E_1 or for E_2 make jointly a necessary condition for the pair (E_1, E_2) to be weakly h.s. (**P 1136**)

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