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ON A CERTAIN NON-LINEAR EQUATION

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In the mathematical theory of the water percolation, the problem of the infiltration of water from a cylindrical reservoir has been considered (cf., e.g., [1] and the bibliography therein). The investigation of a class of the solutions of this problem leads to the one-dimensional integral equation of the form

$$u^2 = K*u+F. \tag{1}$$

The kernel K is a non-decreasing function vanishing on the left half-line with a jump at the origin. The function F is specified in the sequel and is given by the physical interpretations of the considered problem. Only non-negative solutions u of (1) are of interest, since from a physical point of view u is a surface of the percolating water.

This paper is a continuation of [4] and [5] in two directions. First, we allow K to be a non-negative Radon measure with an atom at the origin and vanishing on the left half-line. Second, we replace u^2 by $G\circ u$, where G is a function defined on R_+ and having a strictly increasing derivative with

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{x\to+\infty} G'(x) = +\infty.$$

There is a vast literature concerning non-linear integral equations, notably [2] and [3] by Krasnosel'skij.

From now on we consider the equation

$$G\circ u = K*u+F \tag{2}$$

with given F, G and K.

We suppose that

(i) $K = c\delta+\mu$, where $c > 0$, δ is the Dirac measure and μ is a non-negative Radon measure such that $\mu\big(R\setminus(0, +\infty)\big) = 0$;

(ii) G is a function defined on $R_+ = [0, +\infty)$, and G' is a continuous strictly increasing function on R_+ such that

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{x\to+\infty} G'(x) = +\infty;$$

(iii) F is a non-decreasing function such that $F(x) = 0$ for $x \leqslant 0$ (may be $F \equiv 0$).

THEOREM 1. *Equation* (2) *has precisely one solution in the class* M *of locally bounded Borel functions on* R, *which are positive on* $R_+ \setminus \{0\}$ *and vanish outside. Moreover, the unique solution is non-decreasing.*

For the proof of Theorem 1 we need 3 simple lemmas.

First we note that

$$L(x) = \begin{cases} 0 & \text{for } x = 0, \\ x^{-1}G(x) & \text{for } x > 0 \end{cases}$$

is a strictly increasing function on R_+, differentiable on $R_+ \setminus \{0\}$ and having the inverse function L^{-1} which, of course, is defined on the whole R_+.

LEMMA 1. *If* $u \in M$ *is a solution of* (2), *then*

$$L^{-1}(c) \leqslant u(x) \leqslant L^{-1}\big(c + \mu((0, x]) + [L^{-1}(c)]^{-1}F(x)\big) \quad \textit{for } x > 0. \tag{3}$$

In fact, since $G(u(x)) \geqslant cu(x)$, we get

$$[u(x)]^{-1}G(u(x)) \geqslant c \quad \text{for } x > 0,$$

and so the left-hand side of (3) holds. Let

$$\varphi(x) = \sup_{s \in (0, x]} u(s).$$

We have

$$G(u(s)) \leqslant \varphi(x)\big[c + \mu((0, s])\big] + F(s) \quad \text{for } s \in (0, x].$$

Hence

$$G(\varphi(x)) \leqslant \varphi(x)\big[c + \mu((0, x])\big] + F(x)$$

and

$$[\varphi(x)]^{-1}G(\varphi(x)) \leqslant c + \mu((0, x]) + [\varphi(x)]^{-1}F(x).$$

Therefore, by the left-hand side of (3), we have

$$[\varphi(x)]^{-1}G(\varphi(x)) \leqslant c + \mu((0, x]) + [L^{-1}(c)]^{-1}F(x) \quad \text{for } x > 0,$$

which is the right-hand side of (3).

LEMMA 2. *The function*

$$H(x) = G(x) - cx \quad \textit{for } x \in [L^{-1}(c), +\infty)$$

has the inverse function H^{-1} *which is defined in* R_+.

This follows from $H(L^{-1}(c)) = 0$ and from the fact that $H'(x) = G'(x) - c$ is positive for $x \geqslant L^{-1}(c)$.

LEMMA 3. *For every* $b > 0$ *there exists a* $\beta_0 > 0$ *such that*

$$\int_{[0,b]} e^{-\beta s} d\mu(s) \leqslant \frac{1}{2}\big[G'(L^{-1}(c)) - c\big] \quad \textit{for } \beta \geqslant \beta_0.$$

Proof of Theorem 1. Existence. Let

$$\underline{f}(x) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ L^{-1}(c) & \text{for } x > 0, \end{cases}$$

and

$$\bar{f}(x) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ L^{-1}\big(c + \mu((0, x]) + [L^{-1}(c)]^{-1} F(x)\big) & \text{for } x > 0. \end{cases}$$

Consequently $\underline{f}, \bar{f} \in M$. For $f \in M$, by Lemma 2, we define the operator T as

$$T(f)(x) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ H^{-1}\big((\mu * f)(x) + F(x)\big) & \text{for } x > 0, \end{cases} \tag{4}$$

and we see that

$$T(\underline{f})(x) \geqslant \underline{f}(x) \tag{5}$$

and

$$T(\bar{f})(x) \leqslant \bar{f}(x) \quad \text{for all } x.$$

For every natural number n the function $T^n(\underline{f})$ is non-decreasing. Hence $T^n(\underline{f}) \in M$. We consider the following sequence of functions:

$$u_1 = \underline{f}, \quad u_{n+1} = T(u_n). \tag{6}$$

For $f_1, f_2 \in M$ satisfying the inequality $f_1(x) \leqslant f_2(x)$ for all $x \in R$ we have $T(f_1)(x) \leqslant T(f_2)(x)$ for all $x \in R$. Hence, by (5) and (6), we get $u_{n+1}(x) \geqslant u_n(x)$ for all $x \in R$. Moreover, by the inequality

$$\underline{f}(x) \leqslant T(\underline{f})(x) \leqslant T(\bar{f})(x) \leqslant \bar{f}(x),$$

we obtain

$$u_n(x) \leqslant \bar{f}(x) \quad \text{for every natural } n \text{ and } x \in R;$$

consequently, the sequence $u_n(x)$ is convergent for every $x \in R$. Let

$$u(x) = \lim_{n \to \infty} u_n(x).$$

It is clear that u is non-decreasing and belongs to M. By (4) and (6), u satisfies (2).

Uniqueness. Suppose first that the non-decreasing function

$$r(x) = L^{-1}\big(c + \mu((0, x]) + [L^{-1}(c)]^{-1} F(x)\big) - L^{-1}(c)$$

is positive for $x > 0$. Let $u, \bar{u} \in M$ be two solutions of (2) and let b be a positive number. For $x \in (0, b]$ we have

$$|u(x) - \bar{u}(x)| \leqslant \big[G'\big(L^{-1}(c)\big) - c\big]^{-1} d_b(u, \bar{u}) \int_{[0,x]} e^{\beta(x-s)} r(x-s)\, d\mu(s), \tag{7}$$

where

$$d_b(u, \bar{u}) = \sup_{s\in(0,b]} e^{-\beta s}[r(s)]^{-1}|u(s)-\bar{u}(s)| \tag{8}$$

and β is a positive number. From (7) we obtain the following inequality:

$$|u(x)-\bar{u}(x)| \leqslant \left[G'\big(L^{-1}(c)\big)-c\right]^{-1} d_b(u, \bar{u}) e^{\beta x} r(x) \int_{[0,x]} e^{-\beta s} d\mu(s). \tag{9}$$

From (9), by Lemma 3, we obtain

$$|u(x)-\bar{u}(x)| \leqslant \frac{1}{2} d_b(u, \bar{u}) e^{\beta x} r(x) \quad \text{ for } \beta \geqslant \beta_0.$$

We get $d_b(u, \bar{u}) = 0$, so, by (8),

$$\sup_{s\in(0,b]} |u(s)-\bar{u}(s)| = 0$$

and the uniqueness follows.

If, however, $r(x) = 0$ for $x \in (0, a]$, then $u(x) = L^{-1}(c)$ for $x \in (0, a]$.

THEOREM 2. *If μ is absolutely continuous with respect to the Lebesgue measure and F is continuous, then the solution $u \in M$ of* (2) *is continuous on $R_+\setminus\{0\}$.*

Proof. If μ is absolutely continuous with respect to the Lebesgue measure, then, by the Radon-Nikodym theorem, equation (2) can be written in the form

$$u(x) = H^{-1}\Big(\int_{[0,x]} u(x-s)g(s)ds + F(x)\Big) \quad \text{ for } x > 0, \tag{10}$$

where g is a non-negative Lebesgue locally integrable function. Since $u \in M$ is locally integrable, the function

$$\int_{[0,x]} u(x-s)g(s)ds$$

is continuous for $x > 0$. Consequently, by (10), u is continuous on $R_+\setminus\{0\}$.

The next theorem gives a dependence of the solution of (2) on F. We suppose that functions F_j $(j = 1, 2)$ satisfy (iii) and that one of these functions is positive on $R_+\setminus\{0\}$. Let $u_j \in M$ be the solution of the equation

$$G(u) = (c\delta+\mu)*u+F_j \quad (j = 1, 2).$$

Let

$$\bar{F}(x) = \max\big(F_1(x), F_2(x)\big)$$

and

$$w(x) = L^{-1}\Big(c+\mu\big((0, x]\big)+[L^{-1}(c)]^{-1}\bar{F}(x)\Big)-L^{-1}(c).$$

THEOREM 3. *For* $b > 0$

$$\sup_{s\in(0,b]} |u_1(s)-u_2(s)| \leqslant 2\left[G'\left(L^{-1}(c)\right)-c\right]^{-1}\exp[\beta_0 b]\, w(b)\sup_{s\in(0,b]}[w(s)]^{-1}|F_1(s)-F_2(s)|, \tag{11}$$

where β_0 *is as in Lemma* 3.

Proof. As in the proof of the uniqueness we have

$$|u_1(x)-u_2(x)| \leqslant \frac{1}{2}\exp[\beta_0 x]w(x)\bar{d}_b(u_1,u_2)+\left[G'\left(L^{-1}(c)\right)-c\right]^{-1}|F_1(x)-F_2(x)| \tag{12}$$

for $x\in(0,b]$, where

$$\bar{d}_b(u_1,u_2) = \sup_{s\in(0,b]}\exp[-\beta_0 s][w(s)]^{-1}|u_1(s)-u_2(s)|.$$

Since $\bar{d}_b(F_1,F_2)$ is finite, from (12) we get

$$\bar{d}_b(u_1,u_2)\leqslant\frac{1}{2}\bar{d}_b(u_1,u_2)+\left[G'\left(L^{-1}(c)\right)-c\right]^{-1}\bar{d}_b(F_1,F_2),$$

and hence

$$\bar{d}_b(u_1,u_2)\leqslant 2\left[G'\left(L^{-1}(c)\right)-c\right]^{-1}\bar{d}_b(F_1,F_2),$$

which, by the inequalities

$$\exp[-\beta_0 b][w(b)]^{-1}\sup_{s\in(0,b]}|u_1(s)-u_2(s)|\leqslant\bar{d}_b(u_1,u_2)$$

and

$$\bar{d}_b(F_1,F_2)\leqslant\sup_{s\in(0,b]}[w(s)]^{-1}|F_1(s)-F_2(s)|$$

implies (11).

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