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# EVERY PARACOMPACT $C^m$-MANIFOLD MODELLED ON THE INFINITE COUNTABLE PRODUCT OF LINES IS $C^m$-STABLE

BY

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Let $M$ be a $C^m$-manifold modelled on a linear metric space $E$. We say that $M$ is *$C^m$-stable* ($m \geqslant 0$) if there exists a $C^m$-isomorphism(¹) of $M$ onto $M \times E$. The $C^0$-stability of manifolds is also referred to as the topological stability, and the $C^m$-stability with $m > 0$ as the differential stability.

The topological stability of manifolds modelled on infinite dimensional separable Fréchet spaces was established by Anderson and Schori [2], and this result has been later extended by Schori [9] to a wider class of models including non-separable Hilbert spaces. The $C^\infty$-stability of all Hilbert manifolds, and also some Banach manifolds, is the result of combined efforts of Burghelea and Kuiper [4], Moulis [8], Eells and Elworthy [5] and Elworthy [6]. It seems, however, that their technique is not applicable in the case of differentiable manifolds modelled on Fréchet spaces(²), which do not admit continuous norms. The simplest space of this kind is $s$, the infinite countable product of lines.

In this paper we establish the differential stability of paracompact manifolds modelled on $s$, using an argument which is a suitable adaptation of the Anderson and Schori method. The theorem is stated and proved in Section 4, in foregoing sections preparatory constructions are performed.

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(¹) By a *$C^m$-isomorphism* for $m \geqslant 1$ we mean a diffeomorphism of class $C^m$, and $C^0$-isomorphisms are homeomorphisms.

(²) It is not commonly agreed what is the natural notion of differentiability in a Fréchet space; the notion of differentiability we use is defined in Section 1.

**1. Preliminaries.** Let $R$ denote the real line regarded as a metric space with the metric

$$\delta(t, t') = |t-t'|/(1+|t-t'|).$$

By $N$ we denote the set of positive integers; $[1, \infty]$ is the one-point compactification of the half-line $\{t \in R: t \geqslant 1\}$. For any topological space $Y$, we denote by $Y^N$ the product of $\aleph_0$ copies of $Y$ labelled by positive integers.

Let $s = R^N$ be the Fréchet space of all real sequences $x = (x(n))$ with the coordinate-wise linear operations and with the topology defined by the system of pseudo-norms

$$|x|_i = |x(1)| + \ldots + |x(i)| \quad (i = 1, 2, \ldots).$$

We use the following pseudo-metrics on $s$:

$$d_n(x, x') = \sum_{i=1}^{\infty} 2^{-i}\delta(x(i), x'(i)) \quad (n = 1, 2, \ldots, \infty).$$

Obviously, $d_\infty$ is a metric compatible with the topology of $s$.

We also deal with the product space $s^N$; vectors of $s^N$ are sequences $y = (y_n)$, where $y_n = (y_n(i)) \in R^N = s$. Clearly, $s$ and $s^N$ are linearly homeomorphic; by the *standard isomorphism* from $s$ onto $s^N$ we mean the map $\sigma$ defined by

$$\sigma(x) = (y_n), \quad \text{where } y_n(i) = x(2^{n-1}(2i-1)).$$

For any $k \in N$, we define

$$p_k: s^N \to s^k \quad \text{and} \quad \pi_k: s \to R^k$$

by the formulas

$$p_k(y) = (y_1, \ldots, y_k) \quad \text{for } y = (y_i) \in s^N$$

and

$$\pi_k(x) = (x(1), \ldots, x(k)) \quad \text{for } x = (x(i)) \in s,$$

respectively.

Let $y = (y_n) \in s^N$ and $k \in N$. Then $y_k$ is called the *k-th vector coordinate* of $y$, and the $x(k)$ with $x = \sigma^{-1}(y)$ is called the *k-th scalar coordinate* of $y$.

For every $y, y' \in s^N$, $n \in N \cup \{\infty\}$, we write

$$d_n(y, y') = d_n(\sigma^{-1}(y), \sigma^{-1}(y')). \tag{1}$$

Let $X$ be a Fréchet space (i.e. a locally convex complete linear topological space). A set $\mathscr{A}$ of pseudo-norms for $X$ is said to be *fundamental* if the family $\{\{x \in X: a(x) < \varepsilon\}: a \in \mathscr{A}, \varepsilon > 0\}$ is a base of neighbourhoods of zero in $X$.

In this paper we use the following notion of differentiability:

Definition 1. Let $E$ and $F$ be Fréchet spaces with fundamental systems of pseudonorms $\mathscr{A}$ and $\mathscr{B}$, respectively; let $U$ be an open subset of $E$ and let $m \in N \cup \{\infty\}$. A map $f\colon U \to F$ is said to be of *class* $C^m$ if, for every $x \in U$ and $k < m+1$, there is a continuous $k$-linear operator $a_k(x)\colon E^k \to F$ such that for each $\beta \in \mathscr{B}$ there exists $a \in \mathscr{A}$ with the property

$$\lim_{h\to 0} \frac{1}{a(h)} \beta\Big(f(x+h)-f(x)-\sum_{k<m+1} a_k(x)h^k\Big) = 0,$$

where $h^k = (h, \ldots, h) \in E^k$, and such that the map

$$(h_1, \ldots, h_k, x) \to a_k(x)(h_1, \ldots, h_k)$$

is continuous on $E^k \times U$ for each $k < m+1$.

If $A$ is an arbitrary subset of $E$ and $f\colon A \to F$, then $f$ is said to be of *class* $C^m$ provided that it extends to a map of class $C^m$ on an open neighbourhood of $A$. By a *$C^m$-diffeomorphism* we mean a one-to-one map of class $C^m$ whose inverse is also of class $C^m$.

Remark 1. It is clear that if $f$ is a differentiable real function on $s$, then $f$ depends locally on finite number of coordinates; a map $g\colon U \to s$ is differentiable at a point $y$ if and only if $\pi_k f$ is differentiable at $y$ for $k = 1, 2, \ldots$

Definition 2. Let $V$ and $Z$ be topological spaces and $A$ a closed subset of $V$. Write $(V\times Z)_A = (V\setminus A)\times Z \cup A$; let $p\colon (V\times Z)_A \to V$ be the *collapse map*: $p(v, z) = v$ for $(v, z) \in (V\setminus A)\times Z$ and $p(a) = a$ for $a \in A$. The set $(V\times Z)_A$ will be regarded as a topological space whose topology is determined by the base consisting of all open subsets of $(V\setminus A)\times Z$ and of all sets $p^{-1}(U)$, where $U$ is an open subset of $V$. Spaces $(V\times Z)_A$ will be called *reduced Cartesian products*.

**2. Existence of smooth steering functions.** Any continuous function $\lambda\colon s \to [1, \infty]$ such that $\lambda(x) = \lambda(x')$, whenever $\lambda(x) \leqslant n$ and $\pi_n(x-x') = 0$, is called *steering*. The steering function $\lambda$ is said to be *smooth* if $\lambda | s\setminus\lambda^{-1}(\infty)$ is of class $C^\infty$. A function $\varrho\colon s^N \to [1, \infty]$ is called *steering* provided that $\varrho\circ\sigma$ is a steering function on $s$.

A *steering triple* is any triple $(W, \varrho, \lambda)$, where $W$ is an open subset of $s^N$, and $\varrho$ and $\lambda$ are steering functions on $s^N$ such that

(1) $\varrho(y) = \lambda(y) = \infty$ for all $y \in s^N\setminus W$;

(2) if $A = \varrho^{-1}(\infty)\cap W$ and $B = \lambda^{-1}(\infty)\cap W$, then $A \subset B$ and $\overline{B\setminus A} \cap W = B\setminus A$;

(3) $2^{\varrho(y)} \geqslant 4/d_\infty(y, s^N\setminus W)$ for all $y \in W$;

(4) $\varrho | L = \lambda | L$, where $L = \{y \in s^N\colon d_\infty(y, s^N\setminus W) \leqslant d_\infty(y, \overline{B\setminus A})\}$.

The symbol $d_\infty(\cdot,\cdot)$ in formulas (3) and (4) denotes the distance between a point and a set induced by the metric $d_\infty$ on $s^N$ defined by (1) from Section 1.

PROPOSITION 1. *If $\lambda$ is a smooth steering function on $s$ and $A$ is a closed subset of $s$, then there exists a smooth steering function $\varrho$ such that $\varrho|A = \lambda|A$ and $\varrho(x) < \infty$ for all $x \in s \setminus A$. If $A$ is a closed subset of $s$, then there exists a smooth steering function $\mu$ on $s$ such that $2^{\mu(x)} \geqslant 4/d_\infty(x, A)$ for every $x \in s$, and $\mu^{-1}(\infty) = A$.*

Proof. For any $n \in N$, $\delta > 0$ and $D \subset s$, write

$$O_n(D, \delta) = \{x \in s : d_n(x, D) < \delta\}.$$

Now, we define the sets

$$X_n = \{x \in s : \lambda(x) \leqslant n\},$$

$$U_n = X_{n+1} \cap \big(s \setminus O_n(A, 1/n)\big) \quad \text{and} \quad A_n = A \cap X_n.$$

It is easy to see that, for every $n \in N$,

$$A_n \subset A_{n+1}, \quad U_n = \bar{U}_n \subset \operatorname{Int} U_{n+1} \quad \text{and} \quad A_{n+1} = \bar{A}_{n+1} \subset U_{n+1}.$$

We construct, by induction, a sequence of $C^\infty$-maps

$$\varrho_n : O_n(U_n, 1/2n) \to [1, n+2]$$

which have the following properties:

($a_n$) $\varrho_n(x) = \lambda(x)$ for $x \in O_n(A_{n+1}, 1/2n)$,

($b_n$) $\varrho_n(x) = \varrho_k(x)$ for $x \in U_k$ with $k \leqslant n$,

($c_n$) $\varrho_n(x) \geqslant n$ for $x \in U_n \setminus U_{n-1}$,

($d_n$) $\varrho_n(x) \geqslant n+1$ for $x \in O_n(U_n, 1/2n) \setminus U_n$,

($e_n$) $\varrho_n(x) = \varrho_n(x')$ if $\pi_n(x) = \pi_n(x')$.

Let $f_1 : R \to [0, 1]$ be a function of class $C^\infty$ such that

$$\pi_1\big(O_1(A_2, 1/2)\big) \subset f_1^{-1}(1) \quad \text{and} \quad \operatorname{supp} f_1 \subset \pi_1\big(O_1(A_2, 1)\big).$$

For all $x \in O_1(U_1, 1/2)$, we write

$$\varrho_1(x) = f_1\big(\pi_1(x)\big)\lambda(x) + 2\Big(1 - f_1\big(\pi_1(x)\big)\Big).$$

Then the function $\varrho_1$ satisfies conditions ($a_1$)-($e_1$).

Suppose now that we have constructed functions $\varrho_k$ with properties ($a_k$)-($e_k$) for all $k \leqslant n-1$. Let $f_n : R^n \to [0, 1]$ be a function of class $C^\infty$ such that

$$\pi_n\big(O_n(A_{n+1} \cup U_{n-1}, 1/2n)\big) \subset f_n^{-1}(1)$$

and

$$\operatorname{supp} f_n \subset \pi_n\Big(O_n\big(A_{n+1} \cup U_{n-1}, 1/(2n-2)\big)\Big).$$

The function

$$g_n: O_n(A_{n+1} \cup U_{n-1}, 1/(2n-2)) \to [1, n+1],$$

given by

$$g_n(x) = \begin{cases} \lambda(x) & \text{for } x \in O_n(A_{n+1}, 1/(2n-2)), \\ \varrho_{n-1}(x) & \text{for } x \in O_n(U_{n-1}, 1/(2n-2)), \end{cases}$$

is well defined and continuous by virtue of $(a_{n-1})$. Next, for $x \in O_n(U_n, 1/2n)$, we let

$$\varrho_n(x) = \begin{cases} g_n(\pi_n(x))f_n(x) + (n+1)\big(1 - f_n(\pi_n(x))\big) & \\ & \text{for } x \in O_n(A_{n+1} \cup U_{n-1}, 1/(2n-2)), \\ n+1 & \text{for } x \notin O_n(A_{n+1} \cup U_{n-1}, 1/(2n-2)). \end{cases}$$

We have $f_n(\pi_n(x)) = 0$ for $x \notin O_n(A_{n+1} \cup U_{n-1}, 1/(2n-2))$. Therefore, the function $\varrho_n$ is continuous. Also it is easy to check that $\varrho_n$ is of class $C^\infty$. Since

$$s \setminus \bigcup_{n \in N} U_n = \{x \in A: \lambda(x) = \infty\},$$

the function $\varrho$, defined by

$$\varrho(x) = \begin{cases} \varrho_n((x)) & \text{for } x \in U_n, \\ \infty & \text{for } x \in s \setminus \bigcup_{n \in N} U_n, \end{cases}$$

is of class $C^\infty$ and it is an extension of the function $\lambda | A$. From properties $(a_k)$-$(e_k)$ it follows that $\varrho$ is a steering function. This completes the proof of the first assertion of the proposition.

To establish the second assertion we let $\mu = 2 + \varrho$, where $\varrho$ is the function constructed above for the set $A$ and for $\lambda \equiv \infty$.

**3. The displacement homotopy and partial flattening maps.** In this section we shall introduce certain auxiliary maps of reduced Cartesian products and establish some properties of these maps.

Definition 3. A homotopy $f_t: s^N \times s \to s^N$, $1 \leqslant t \leqslant \infty$, is called a *smooth displacement homotopy* if

(1) the map $p_k(f_t(y, x)) = p_k(y)$ for $k \in N$, $t \geqslant k$, and $f_\infty(y, x) = y$ for all $(y, x) \in s^N \times s$,

and

(2) the map $f(y, x, t) = (f_t(y, x), t)$ is a $C^\infty$-diffeomorphism of $s^N \times s \times$ $\times [1, \infty]$ onto $s^N \times [1, \infty]$.

The existence of smooth displacement homotopies is asserted by Lemma 1.

Suppose that $\varrho\colon s^N\to[1,\infty]$. If $A=\varrho^{-1}(\infty)$ and $\{f_t\}_{1\leqslant t\leqslant\infty}$ is a displacement homotopy, then let

$$f_\varrho\colon (s^N\times s)_A\to s^N$$

be the map defined by the formulas

(3) $\quad f_\varrho(y,x)=f_{\varrho(y)}(y,x)$ for $y\notin A$, and $f_\varrho(y)=y$ for $y\in A$.

Suppose that $(W,\varrho,\lambda)$ is a steering triple, $A=W\cap\varrho^{-1}(\infty)$, and $B=W\cap\lambda^{-1}(\infty)$. The fundamental role in the proof of the stability theorem is played by the following partial flattening maps:

$$f_\lambda^{-1}\circ f_\varrho\colon (W\times s)_A\to(W\times s)_B.$$

Differentiability properties of these maps are discussed in Lemmas 2-5.

LEMMA 1. *There exists a smooth displacement homotopy* $\{f_t\}_{1\leqslant t\leqslant\infty}$ *on the space* $s^N\times s$.

Proof. Let $a\colon R\to[0,1]$ be a function of class $C^\infty$ such that $a(t)=0$ for $t\leqslant 0$, and $a(t)=1$ for $t\geqslant 1$; let $b(t)=1-a(t)$ for $t\in R$. We write $f_\infty(y,x)=y$ and, for $n\leqslant t<n+1$,

$$f_{n+t}(y,x)$$
$$=\big(y_1,\ldots,y_n,b(t)x+a(t)y_{n+1},a(t)x-b(t)y_{n+1},-y_{n+2},-y_{n+3},\ldots\big).$$

It is easy to see that the homotopy $\{f_t\}_{1\leqslant t\leqslant\infty}$ has the required properties.

LEMMA 2. *Let $\varrho$ be a smooth steering function on $s^N$ and let $A=\varrho^{-1}(\infty)$. Then $F=f_\varrho$ is a homeomorphism of $(s^N\times a)_A$ onto $s^N$ and has the properties*

(4) $\quad F(z)=z$ *for* $z\in A$,

(5) $\quad p_kF(y,x)=p_k(y)$ *for* $k\leqslant\varrho(y)$,

(6) $\quad F\mid(s^N\setminus A)\times s$ *is a* $C^\infty$*-diffeomorphism.*

Proof. The map $F$ is continuous. In fact, if $y^k\in s^N$ with $\lim_k y^k=y\in\operatorname{bd}A$ and $x^k$ are arbitrary points of $s$, then

$$\lim_k F(y^k,x^k)=\lim_k f_{\varrho(y^k)}(y^k,x^k)=y.$$

By condition (1) and by the definition of steering functions, we have $\varrho\big(F(y,x)\big)=\varrho(y)$ for all $(y,x)\in(s^N\setminus A)\times s$. Therefore, the map $F^{-1}\colon s^N\to(s^N\times s)_A$ exists and is given by the formula

$$F^{-1}(y)=\begin{cases} y & \text{for } y\in A,\\ f^{-1}\big(y,\varrho(y)\big) & \text{for } y\in s^N\setminus A.\end{cases}$$

Now, by the definition of the reduced product topology, it follows that $F^{-1}$ is continuous. Finally, the restrictions $F|(s^N \setminus A)\times s$ and $F^{-1}|s^N\setminus A$, being compositions of $C^\infty$-maps, are of class $C^\infty$.

LEMMA 3. *Suppose that $\mu_1$ and $\mu_2$ are smooth steering functions, and*

$$A_j = \mu_j^{-1}(\infty), \qquad F_j = f_{\mu_j}\colon (s^N\times s)_{A_j}\to s^N \quad \textit{for } j = 1, 2;$$

*$U$ is an open subset of $s^N$, and $\varphi$ is a $C^m$-diffeomorphism of $U$ into $s^N$ such that*

$$\varphi(A_1\cap U) = A_2\cap\varphi(U).$$

*Let*

$$\tilde{\varphi}\colon (U\times s)_{A_1\cap U}\to \big(\varphi(U)\times s\big)_{A_2\cap\varphi(U)}$$

*be the map defined by the formula*

$$\tilde{\varphi}(z) = \begin{cases} \big(\varphi(y), x\big) & \textit{if } z = (y, x)\in(U\setminus A_1)\times s, \\ \varphi(z) & \textit{if } z\in A_1. \end{cases}$$

*Then the map $G = F_2\circ\tilde{\varphi}\circ F_1^{-1}|F_1\big((U\times s)_{A_1\cap U}\big)$ is a $C^m$-diffeomorphism of $F_1\big((U\times s)_{A_1\cap U}\big)$ onto an open subset of $s^N$.*

Proof. By Lemma 2, $F_1$ and $F_2$ are homeomorphisms. Therefore, $G$ is a homeomorphism. Observe that $(U\setminus A_1)\times s$ and $\big(\varphi(U)\setminus A_2\big)\times s$ are open subsets of $(s^N\setminus A_1)\times s$ and $(s^N\setminus A_2)\times s$, respectively. Hence $F_1|(U\setminus A_1)\times s$ and $F_2|\big(\varphi(U)\setminus A_2\big)\times s$ are $C^m$-diffeomorphisms. Clearly,

$$(\varphi\times \mathrm{id})|(U\setminus A_1)\times s = \tilde{\varphi}|(U\setminus A_1)\times s$$

is a $C^m$-diffeomorphism. Therefore, $G|F_1\big((U\setminus A_1)\times s\big)$ is a $C^m$-diffeomorphism.

Moreover, $F_1(z) = z$ for $z\in A_1\cap U$ and $F_2(z) = z$ for $z\in A_2\cap\varphi(U)$. Thus $G(y) = \varphi(y)$ for $y\in A_1\cap U$, whence $G|\mathrm{Int}(A_1\cap U)$ is a $C^m$-diffeomorphism.

Now, assume that $\bar{y}\in \mathrm{bd}\,A_1$. Then

(*) for every $k\in N$, there is a neighbourhood $U_k\ni\bar{y}$ such that $\pi_k G(y) = \pi_k(y)$ for $y\in U_k$.

This is a consequence of the fact that

$$\lim_{y\to\bar{y}} \mu_1\big(F_1(y)\big) = \lim_{y\to\bar{y}} \mu_2\big(\tilde{\varphi}F_1(y)\big) = \infty$$

and that every scalar coordinate of a differentiable map on $s^N$ depends locally on a finite number of scalar coordinates of the points.

Using property (*) and the fact that $\varphi$ is a $C^m$-diffeomorphism, one easily checks that, for every $k < m+1$, the $k$-linear operators $a_k(y)$ of Definition 1 exist for all $y$ in a neighbourhood of $\bar{y}$ and depend continuously on $y$.

The same argument can be applied to establish the differentiability of the map $G^{-1}$.

LEMMA 4. *Suppose that* $(W, \varrho, \lambda)$ *is a steering triple,* $A = \varrho^{-1}(\infty)\cap W$ *and* $B = \lambda^{-1}(\infty)\cap W$. *Then the map*

$$H = f_\lambda^{-1}\circ f_\varrho|(W\times s)_A$$

*is a homeomorphism of* $(W\times s)_A$ *onto* $(W\times s)_B$ *and satisfies the conditions*

(7) $H(z) = z$ *if* $p(z)\in A\cup(W\setminus C)$, *where* $C$ *is a closed subset of* $s^N$ *such that* $\overline{B\setminus A}\subset C\subset W$;

(8) $H|(W\setminus A)\times s\setminus H^{-1}(B)$ *is a* $C^\infty$*-diffeomorphism*;

(9) $H|\mathrm{Int}\,H^{-1}(B)\setminus A$ *is a* $C^\infty$*-diffeomorphism.*

Proof. By Lemma 2, $f_\lambda^{-1}\circ f_\varrho$ is a homeomorphism of $(s^N\times s)_{A\cup(s^N\setminus W)}$ onto $(s^N\times s)_{B\cup(s^N\setminus W)}$. Moreover, $f_\varrho$ carries $(W\times s)_A$ onto $W$, and $f_\lambda$ carries $(W\times s)_B$ onto $W$. Therefore, $H$ is a homeomorphism of $(W\times s)_A$ onto $(W\times s)_B$.

Conditions (8) and (9) follow immediately from Lemma 2.

To establish (7), write

$$C = \{y\in s^N\colon d_\infty(y, \overline{B\setminus A})\leqslant 4d_\infty(y, s^N\setminus W)\}.$$

If $z\in(W\times s)_A$ and $p(z)\in W\setminus C$, then

$$d_\infty\big(p(z), s^N\setminus W\big) < \frac{1}{4}d_\infty\big(p(z), \overline{B\setminus A}\big).$$

By condition (3) of Section 2 and by property (5) of Lemma 2, we have

$$d_\infty\big(f_\varrho(z), p(z)\big)\leqslant\frac{1}{2}d_\infty\big(p(z), s^N\setminus W\big).$$

Hence,

$$f_\varrho(z)\in L = \{y\in s^N\colon d_\infty(y, s^N\setminus W)\leqslant d_\infty(y, \overline{B\setminus A})\}.$$

Thus, by condition (4) of Section 2,

$$\lambda\big(p(f_\varrho(z))\big) = \varrho\big(p(f_\varrho(z))\big)$$

and this gives

$$H(z) = f_\lambda^{-1}\circ f_\varrho(z) = f_\varrho^{-1}\circ f_\varrho(z) = z.$$

**4. Local flattening of manifolds. The stability theorem.**

Definitions. Assume $M$ is a $C^m$-manifold modelled on a Fréchet space $Y$, i.e. $M$ is a paracompact topological space equipped with a differentiable structure given by an atlas $\{U_\alpha, \varphi_\alpha\}_{\alpha\in\mathscr{A}}$, where $\{U_\alpha\}_{\alpha\in\mathscr{A}}$ is an open cover of $M$, and, for each $\alpha \in \mathscr{A}$, $\varphi_\alpha: U_\alpha \to Y$ is an open embedding such that $\varphi_\beta \circ \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha \cap U_\beta) \to Y$ is a $C^m$-diffeomorphism.

A pair $(V, \psi)$ is said to be a *regular chart* if $\psi: V \to Y$ is a $C^m$-diffeomorphism onto an open subset of $Y$ and $\psi$ extends to a closed embedding of $\bar{U}$ into $Y$. A collection of regular charts $\{V_\beta, \psi_\beta\}_{\beta \in \mathscr{B}}$ such that $\{V_\beta\}_{\beta \in \mathscr{B}}$ is a cover for $M$ is said to be a *regular atlas* for $M$.

A cover of $M$ is called *star-finite* if each member of it intersects only finitely many members of the cover. An open cover $\{V_\beta\}_{\beta \in \mathscr{B}}$ is a *refinement* of the cover $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ if each $V_\beta$ is contained in some $U_{\alpha(\beta)}$. An open cover $\{V_\alpha\}_{\alpha \in \mathscr{A}}$ is a *shrunk-refinement* of the cover $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ if $\bar{V}_\alpha \subset U_\alpha$ for each $\alpha \in \mathscr{A}$.

PROPOSITION 2. *Every connected paracompact $C^m$-manifold* $(0 \leqslant m \leqslant \infty)$ *modelled on the space $s$ admits a countable regular atlas.*

Proof. Since $M$ is connected paracompact locally metrizable and locally separable, it follows that $M$ is metrizable and separable (see [10], p. 111, and [7], Chapter 4, Section 4, Theorem 5). Therefore, for every open cover $\{V_\beta\}_{\beta \in \mathscr{B}}$ of $M$, there is a shrunk-refinement (see [7], Chapter 5, Section 1). Also every cover of $M$ admits a star-finite refinement (see [7], Chapter 5, Section 1, Theorem 4).

Since each point $x \in s$ has a base of open neighbourhoods diffeomorphic to the whole $s$, we conclude that $M$ admits an atlas $\{V_n, \psi_n\}_{n \in N}$ such that $\psi_n(V_n) = s$ (remember that $M$ is separable). Let $\{W_n\}_{n \in N}$ be a shrunk-refinement of $\{V_n\}$. Obviously, $\{W_n, \psi_n | W_n\}_{n \in N}$ is a regular atlas for $M$. Let $\{U_i\}_{i \in N}$ be a star-finite refinement of $\{W_n\}_{n \in N}$ and let $\{U_i, \varphi_i\}_{i \in N}$ be the corresponding atlas, i.e. $\varphi_i = \psi_{n(i)} | U_i$. Thus $\{U_i, \varphi_i\}_{i \in N}$ has the property required in the proposition.

LEMMA 5. *Let $\{U_i, \varphi_i\}_{i \in N}$ be a regular star-finite atlas on a $C^m$-manifold $M$ and let $\{V_i\}_{i \in N}$ be a shrunk-refinement of the cover $\{U_i\}_{i \in N}$. Write*

$$W_i = \bigcup_{j \leqslant i} V_j \quad \text{for } i \in N \text{ and } W_0 = \emptyset.$$

*Then there are homeomorphisms $g_i: M \times s \to (M \times s)_{\bar{W}_i}$ such that*

(1) *for each $n$ there exists $k$ with $g_j(z) = g_k(z)$ if $j \geqslant k$ and $p(z) \in U_n$,*

(2) *$g_i | M \times s \setminus g_i^{-1}(\bar{W}_i)$ is a $C^m$-diffeomorphism,*

(3) *$g_i | g_i^{-1}(W_i)$ is a $C^m$-diffeomorphism for all $i \in N$.*

Proof. Let us take a family of steering triples $\{(\varphi_i(U_i), \varrho_i, \lambda_i)\}_{i \in N}$ such that

$$\varphi_i(\bar{W}_{i-1} \cap U_i) = \varrho_i^{-1}(\infty) \cap \varphi_i(U_i) \quad \text{and} \quad \varphi_i(\bar{W}_i \cap U_i) = \lambda_i^{-1}(\infty) \cap \varphi_i(U_i)$$

(steering functions satisfying these conditions exist by Proposition 1). By Lemma 4, each map

$$H_i = f_{\lambda_i}^{-1} \circ f_{\varrho_i} | (\varphi_i(U_i) \times s)_{\varphi_i(\overline{W}_{i-1} \cap U_i)}$$

related to the triple $(\varphi_i(U_i), \varrho_i, \lambda_i)$ carries $(\varphi_i(U_i) \times s)_{\varphi_i(\overline{W}_{i-1} \cap U_i)}$ onto $(\varphi_i(U_i) \times s)_{\varphi_i(\overline{W}_i \cap U_i)}$ and has the properties

(4) $H_i(z) = z$ if $p(z) \in \varphi_i(W_{i-1} \cap U_i) \cup (\varphi_i(U_i) \setminus C_i)$, where $C_i$ is a closed set in $s^N$ such that $\varphi_i(\overline{V}_i) \subset C_i \subset \varphi_i(U_i)$;

(5) $H_i | \varphi_i(U_i \setminus \overline{W}_{i-1}) \times s \setminus H_i^{-1}(\overline{V}_i)$ is a $C^m$-diffeomorphism;

(6) $H_i | H_i^{-1}(\varphi_i(V_i \setminus \overline{W}_{i-1}))$ is a $C^m$-diffeomorphism.

Define the maps $h_i \colon (M \times s)_{\overline{W}_{i-1}} \to (M \times s)_{\overline{W}_i}$ by the formula

$$h_i(z) = \begin{cases} \tilde{\varphi}_i^{-1} \circ H_i \circ \tilde{\varphi}_i(z) & \text{if } p(z) \in U_i \times s, \\ z & \text{if } p(z) \notin U_i \times s. \end{cases}$$

Obviously, $h_i$ is a homeomorphism and

(7) $h_i | (M \setminus \overline{W}_{i-1}) \times s \setminus h_i^{-1}(\overline{V}_i)$ is a $C^m$-diffeomorphism,

(8) $h_i | h_i^{-1}(V_i \setminus \overline{W}_{i-1})$ is a $C^m$-diffeomorphism.

We shall show that $g_i = h_i \circ \ldots \circ h_1$ have properties (1)-(3). Condition (1) is a consequence of (7) of Section 3 and of the fact that the cover $\{U_i\}_{i \in N}$ is star-finite. Condition (2) follows from (9) of Section 3 and the fact that $h_{j-1}\big(\ldots\big(h_1(M \times s \setminus g_i^{-1}(\overline{W}_i))\big)\ldots\big)$ is an open subset of the set $(M \setminus \overline{W}_{j-1}) \times s \setminus h_j^{-1}(\overline{V}_j)$ for $j \leqslant i$.

To establish (3) take any point $\bar{z} \in g_i^{-1}(W_i)$. If $g_j(\bar{z}) \notin \operatorname{bd} V_j$ for all $j \leqslant i$, then, by (7) and (8), $g_i$ is a $C^m$-diffeomorphism in a neighbourhood of $\bar{z}$. In the other case, let $k$ denote the smallest integer $j \leqslant i$ such that $g_j(z) \in \operatorname{bd} V_j$. Then, by (7), $g_{k-1}$ is a $C^m$-diffeomorphism in a neighbourhood of $\bar{z}$. Near the point $g_{k-1}(\bar{z})$ the maps $h_1, \ldots, h_k$ either are equal to the identity or are of the form

$$h_j = \tilde{\varphi}_j^{-1} \circ H_j \circ \tilde{\varphi}_j = \tilde{\varphi}_j^{-1} \circ (f_{\lambda_j}^{-1} \circ f_{\varrho_j}) \circ \tilde{\varphi}_j.$$

Therefore, the map $h_i \circ \ldots \circ h_k$ can be expressed as a composition of maps each of which has one of the forms

$$(9) \qquad \varphi_{\varrho_n} \circ (\tilde{\varphi}_n \circ \tilde{\varphi}_j^{-1}) \circ f_{\lambda_j}^{-1}, \quad f_{\varrho_k} \circ \tilde{\varphi}_k, \quad \tilde{\varphi}_i^{-1} \circ f_{\lambda_i}^{-1}.$$

By Lemma 3, the maps $f_{\varrho_n} \circ (\tilde{\varphi}_n \circ \tilde{\varphi}_j^{-1}) \circ f_{\lambda_j}^{-1}$ are $C^m$-diffeomorphisms in a neighbourhood of the point $f_{\varrho_j}\big(\tilde{\varphi}_j(g_{j-1}(\bar{z}))\big)$. Similarly, the remaining maps in (9) are also diffeomorphisms. Therefore, each $g_i$ is a $C^m$-diffeomorphism in a neighbourhood of $\bar{z}$. Remembering that $\bar{z}$ was an arbitrary point of $g_i^{-1}(W_i)$, we obtain (3). This completes the proof of Lemma 5.

THE STABILITY THEOREM. *Every paracompact $C^m$-manifold $M$ modelled on the space $s$ is $C^m$-diffeomorphic to $M \times s$.*

Proof. Without loss of generality we may assume that $M$ is connected. By Proposition 2, there is a regular star-finite atlas $\{U_i, \varphi_i\}_{i \in N}$ on $M$. By relabelling the cover $\{U_i\}_{i \in N}$, we can make it chain-finite ordered, i.e. there is no infinite subsequence of indices $k_1 < k_2 < k_3 < \ldots$ such that $U_{k_n} \cap U_{k_{n+1}} \neq \emptyset$ for all $n \in N$ (cf. [1], Theorem 2). Let $\{V_i\}_{i \in N}$ be a shrunk refinement of $\{U_i\}$. Write

$$W_i = \bigcup_{j \leqslant i} V_j \quad \text{for } i \in N \text{ and } W_0 = \emptyset.$$

By Lemma 5, we obtain a sequence of homeomorphisms

$$g_i\colon M \times s \to (M \times s)_{\overline{W}_i}.$$

The map

$$g = \lim_i g_i$$

is the required diffeomorphism of $M \times s$ onto $M$. By property (1) and the chain-finiteness of the cover $\{U_i\}$, there is a $k$ such that $g_j(z) = g_k(z)$ for $j \geqslant k$, whenever $p(z) \in U_n$. Therefore, $g$ is well defined and is of class $C^m$. Since every $g_k$ carries $M \times s$ onto $(M \times s)_{\overline{W}_k}$ and since $\{V_i\}$ is an open cover of $M$, we conclude that $g(M \times s) \subset M$ and also we have $x = g(g^{-1}(x))$ for every $x \in V_j$. Since $g^{-1}$ restricted to every open set $U_j$ coincides with $g_j^{-1}$, we conclude that $g$ is onto $M$ and $g^{-1}$ is of class $C^m$.

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