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INDEPENDENCE WITH RESPECT TO FAMILIES OF CHARACTERS

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Introduction. The set of functions

$$f_n(x) = nx \pmod 1, \qquad n = 1, 2, \ldots, \tag{1}$$

defined on $[0, 1)$ and regarded as a subspace of the Tychonoff cube $[0, 1]^{[0,1)}$ becomes dense when projected (by restriction) into $[0, 1]^E$, where $E \subset [0, 1)$ and $E \cup \{1\}$ is independent over the rationals. This is a consequence of the Kronecker multidimensional theorem, according to which if $1, x_1, \ldots, x_r$ are independent over the rationals and $U_1, \ldots, U_r$ are intervals on $[0, 1)$, then there exists an integer n such that

$$nx_1 \in U_1, \ldots, nx_r \in U_r \pmod 1.$$

It was observed by Priestley [7] that if E is, in addition, of positive outer measure, then the sequence $f_n|E$, $n = 1, 2, \ldots$, contains no nontrivial convergent subsequences. Indeed, if $n_0 < n_1 < \ldots$ is a sequence of integers, then according to the Hardy–Littlewood theorem (or an even stronger theorem of Weyl asserting a.e. uniform distribution of $n_k x \pmod 1$) the sequence $n_0 x$, $n_1 x, \ldots \pmod 1$ is dense in $[0, 1)$ for almost all x, and clearly such an x can be found in E, E being not of measure zero. This gives a stengthened version of the known Hewitt–Marczewski–Pondiczery theorem, namely the existence of a countable dense subset having no nontrivial convergent sequences in the Tychonoff cube of weight continuum.

A question arises whether subsequences of the sequence (1) have an analogous property, i.e., whether for any subsequence

$$f_{n_k}(x) = n_k x \pmod 1, \qquad n_0 < n_1 < \ldots,$$

of (1) there exists an uncountable subset E of $[0, 1)$ such that the set of functions $f_{n_k}|E$, $k = 0, 1, \ldots$, is dense in $[0, 1]^E$. In particular, this concerns the sequences

$$f_{p^k}(x) = p^k x \pmod 1, \quad k = 0, 1, \ldots,$$

where p is an integer, $p \geqslant 2$.

A more natural setting of the problem is to consider, instead of (1), the functions

$$(2) \qquad f_n(z) = z^n, \quad n = 0, \pm 1, \pm 2, \ldots,$$

defined on the unit circle $T = \{z: |z| = 1\}$, i.e., the continuous characters of T.

It is known from the work of Hartman and Ryll-Nardzewski [3] that the set of limit functions of a sufficiently thin subsequence of (2) is homeomorphic to the remainder $\beta\omega\setminus\omega$ in the Čech–Stone compactification $\beta\omega$ of the set ω of natural numbers. This is true, in particular, for any sequence

$$(3) \qquad f_{p^k}(z) = z^{p^k}, \quad k = 0, 1, \ldots,$$

where p is an integer, $p \geqslant 2$. More generally, it was shown by Strzelecki [11] that this is the case if the sequence $n_0 < n_1 < \ldots$ is lacunary, i.e., $n_{k+1}/n_k \geqslant \gamma > 1$.

The whole sequence (2) is far from having the property considered above since it is dense in the set bZ of all characters of T and, in particular, dense in itself. In fact, the closure of the sequence (2) in T^T coincides with bZ (see, e.g., [4], Corollary (26.16)). Moreover, bZ is nowhere dense in T^T and it is well known that all the characters except the functions $f_n(z) = z^n$ are nonmeasurable. The corresponding nonmeasurable functions on the unit interval were originally investigated by Sierpiński [9], [10].

In Section 3 it will be proved that for each set

$$\Phi = \{f_{n_k}: n_0 < n_1 < \ldots\}$$

of continuous characters of T there exists a subset E of T of cardinality continuum such that the set

$$\{f_{n_k}|E: k = 0, 1, \ldots\}$$

is dense in T^E. This means that the closure $\bar{\Phi}$ of Φ in T^T (or, equivalently, in bZ) is projected onto T^E (by restriction). In other words, each function $f: E \to T$ extends to a character in $\bar{\Phi} \subset bZ$.

Subsets of T having this property with respect to a given set Φ of continuous characters of T will be called *Φ-independent*. The Φ-independent sets are the smaller the thinner the sets Φ. The problem of the existence of Φ-independent sets which are "large" in the sense of measure or topology is dealt with in Section 3. For

$$\Phi_p = \{f_{p^k}: k = 0, 1, \ldots\}$$

the existence of Φ_p-independent sets of cardinality continuum can be obtained directly by modifying the theorem of Fichtenholz and Kantorovitch [1], but our proof presented in Section 3 is based on a different method and works for arbitrary infinite Φ, answering the question stated at the beginning.

1. Comments concerning the position of Φ_p in bZ. We shall describe in some detail the position, in bZ, of the set Φ_p consisting of the functions (3) as well as the set $\Phi_p^* = \bar{\Phi}_p \setminus \Phi_p$ of limit functions of (3).

Denoting by χ_t the character of Z given by

$$\chi_t(n) = \exp(2\pi int), \qquad n \in Z, \tag{4}$$

consider the set B_p of those functions in bZ which are equal to 1 on all characters χ_{j/p^l}, $l \geq 0, j \in Z$. The set B_p has the structure of a closed subgroup of bZ (in fact, the annihilator of the p^l-roots of unity, $l \geq 0$) and coincides with the intersection of the sets $B_{p,j,l}$, $l \geq 0$, $j \in Z$, consisting of those functions equal to one on a single character χ_{j/p^l}. The function f_n from (2) is in $B_{p,j,l}$ iff $\exp(2\pi ijn/p^l) = 1$. This implies that $f_{p^k} \in B_{p,j,l}$ iff $k \geq l$.

Thus, the set Φ_p^* is contained in B_p and $B_p \cap \Phi_p = \emptyset$. Since B_p is a group, this clearly implies that B_p is nowhere dense in bZ.

The set $B = \bigcap B_p$ of all those functions from bZ which are equal to 1 on all characters (4) with t rational is the connected component of the neutral element of bZ (see, e.g., [4]).

If there exists a prime factor of q which is not a factor of p, then the set B_q does not contain any limit functions of the sequence (3), *i.e.* $B_q \cap \Phi_p^* = \emptyset$.

To see this, project T^T onto the $\chi_{1/q}$-axis. Then all functions in B_q are projected into 1, while the functions f_{p^k} are projected into the set of q-th roots of unity from which 1 is removed (the fraction p^k/q cannot be an integer).

Since $B \subset B_p \cap B_q$ for any two integers p and q, as a consequence of the above observation we have

$$B \cap \Phi_p^* = \emptyset$$

for all integers p, $p \geq 2$.

We recall that a continuous map $g\colon M \to N$ is called *irreducible* if $g(A) = N$ implies $A = M$ for each closed subset of A of M. We note that if E is such that Φ_p^* projects onto T^E (by restriction), then the projection $\pi\colon \Phi_p^* \to T^E$ is far from being irreducible. Indeed, suppose A is a closed subset of Φ_p^* such that $\pi(A) = T^E$ and $\pi|A\colon A \to T^E$ is irreducible (the existence follows from the Zorn lemma). Since T^E is separable, i.e., it has a countable dense subset, the set A is also separable. Thus A is a separable subspace of the space Φ_p^* which is nowhere separable as a homeomorphic copy of $\beta\omega \setminus \omega$. Thus, A is nowhere dense in Φ_p^*.

2. Definition of Φ-independent sets. Let G be an abelian group. A subset E of G is called *independent* if for any distinct elements $x_1, \ldots, x_n$ of E and any integers $k_1, \ldots, k_n$ the equality

$$x_1^{k_1} \ldots x_n^{k_n} = 1$$

can only occur if $k_1 = \ldots = k_n = 0$. This definition implies that an independent set contains no elements of finite order, which makes our notion slightly different from the independence considered in [4] (but consistent with [2], V). Note that $\{x\}$ is independent in G iff x has infinite order.

The independent sets can be characterized by the following extension property:

(i) *E is independent iff any function $f\colon E \to T$ extends to a character on G.*

Indeed, if E is independent, then the formula

$$\chi(x_1^{k_1} \ldots x_n^{k_n}) = f(x_1)^{k_1} \ldots f(x_n)^{k_n}$$

extends f to a character χ on the group generated by E, and next χ can be extended to a character on G (see, e.g., [4], 24.12). The converse is clear.

The following observation will also be useful:

(ii) *If E is independent and $a \neq 1$, then* $\operatorname{card}(E \cap Ea) \leqslant 1$.

To see this suppose $x = ua$, $y = va$, $x \neq y$; $x, y, u, v \in E$. This implies $xy^{-1}vu^{-1} = 1$. If $u \neq y$ and $v \neq x$, then the four elements are distinct and we arrive at a contradiction. If, e.g., $u = y$, then $xvy^{-2} = 1$, which is also impossible.

Throughout the rest of the paper we consider a locally compact abelian group (LCA group) G. By $\hat{G}$ we denote the dual LCA group of all continuous characters on G and by $b\hat{G}$ the group of all characters on G endowed with the compact topology of pointwise convergence inherited from T^G. It is well known that $\hat{G}$ is dense in $b\hat{G}$, the Bohr compactification of $\hat{G}$ ([4], 26.16).

We say that a subset Φ of $\hat{G}$ is *unbounded* if it is not relatively compact for the locally compact (Pontryagin) topology on $\hat{G}$.

DEFINITION. Let $\Phi \subset \hat{G}$ be unbounded. A subset E of G is said to be *Φ-independent* if any function $f\colon E \to T$ extends to a character contained in the (pointwise) closure of Φ in $b\hat{G}$.

A subset of a Φ-independent set is Φ-independent. Also, if $\Psi \subset \Phi \subset \hat{G}$, then every Ψ-independent set is Φ-independent. In particular, any Φ-independent set is independent (see (i)). The converse fails in general.

EXAMPLE 1. Consider

$$\Phi_2 = \{z^{2^n}\colon n = 0, 1, \ldots\} \subset \hat{T}$$

and let

$$z = \exp\left(2\pi i \sum_{n=1}^{\infty} 2^{-n^2}\right).$$

The singleton $\{z\}$ is not Φ_2-independent since $\operatorname{Arg}\varphi(z) < 5\pi/4$ for all $\varphi \in \Phi_2$. On the other hand, $\sum 2^{-n^2}$ is irrational, so z is of infinite order, hence independent in T.

The following example shows that the union of the Φ_p-independent sets (p fixed) in the unit circle is not too large.

EXAMPLE 2. Let $p \geqslant 2$ be an integer. Then there exists a Borel subset A_p of T such that

(a) the Hausdorff dimension of A_p is equal to one,

(b) A_p is disjoint with every Φ_p-independent set.

In fact, let A_p be the set of all $z \in T$ such that the sequence z^{p^n}, $n \geqslant 0$, is not dense in T. Given $\varepsilon > 0$ we fix $k \geqslant 1$ satisfying the condition

$$\log(p^k - 1)/\log p^k > 1 - \varepsilon$$

and define D_p to be the set of all $z = e^{2\pi i t}$ such that $0 < t < 1$ and there is no $2k$-block of zeroes in the expansion of t to the base p,

$$t = \sum_{j=1}^{\infty} t_j/p^j, \qquad 0 \leqslant t_j < p,$$

where in the case of two different representations the one with, say, infinitely many zeroes is chosen. Clearly, $D_p \subset A_p$. We let C_p be the set of all numbers $e^{2\pi i t}$, $0 < t < 1$, containing no block of zeroes of the form

$$t_{nk+1} \ldots t_{(n+1)k} \qquad (n \geqslant 0).$$

We have $C_p \subset D_p$ and C_p can be identified with the set of all numbers in the unit interval in whose p^k-expansion the digit 0 does not occur. Therefore

$$\dim C_p = \log(p^k - 1)/\log p^k > 1 - \varepsilon$$

and, consequently, $\dim A_p = 1$.

The sets C_p belong to the class of Rajchman's (H)-sets, which play an important role in the problem of uniqueness of trigonometric expansions (Rajchman [8]).

3. Existence of Φ-independent sets. Our aim is to obtain uncountable Φ-independent sets. The two methods presented below parallel the construction of independent sets of transitive points in dynamical systems [5]. The first is an application of Mycielski's independence theorem in topological relational structures [6]. Using his theorem, Mycielski obtained algebraically independent sets (a notion slightly different from our $\hat{G}$-independence in the case of abelian groups) of cardinality continuum in any connected locally compact group G, $\operatorname{card} G > 1$ ([6], p. 144). Our case of Φ-independence seems to be less straightforward because of the lack of a purely algebraic condition for Φ-independence. The second method is based on transfinite induction and resembles the well-known construction of the Bernstein set.

In order to obtain uncountable Φ-independent sets for each Φ, we restrict the class of groups by imposing the following condition on G:

(*) For every $m \neq 0$ the mapping $f_m\colon \hat{G} \to \hat{G}$ defined by $f_m(\chi) = \chi^m$ is continuous at infinity.

An equivalent wording is the following:

(**) For every $m \neq 0$ the set $f_m^{-1}(K)$ is compact whenever K is a compact subset of $\hat{G}$.

If G is compact, then $\hat{G}$ is discrete, so "compact" means "finite" in (**). Therefore, for any compact G, (*) is equivalent to

(***) For every $m \neq 0$ there are only finitely many characters of order m in $\hat{G}$.

The class of groups satisfying (*) is vast. First note that every compact monothetic group satisfies (*). In fact, $\hat{G}$ is a subgroup of T ([4], 24.32), so (***) holds. If G is any connected compact abelian group, then $\hat{G}$ is torsion free ([4], 24.25), whence G satisfies (***). It is also easy to see that the class of LCA groups satisfying (**) is closed under direct products. Since (*) clearly holds for R^n, we infer by the structural theorem ([4], 9.14) that any connected LCA group satisfies (*).

Note that if G is compact, then (***) is necessary for the existence of a nonempty Φ-independent set for all unbounded Φ. In fact, suppose

$$\Phi(m) = \{\chi \in \hat{G}\colon \chi^m = 1\}$$

is infinite. Since

$$\operatorname{card}\{\chi(x)\colon \chi \in \Phi(m)\} \leqslant m$$

for each $x \in G$, the singleton $\{x\}$ is never $\Phi(m)$-independent.

We shall prove that (***) is also sufficient for the existence of uncountable Φ-independent sets (Theorems 1 and 2).

From now on, G is an LCA group satisfying (*) and Φ an unbounded subset of $\hat{G}$. We denote by dx the Haar measure on G. The measure of a set $A \subset G$ will be denoted by $|A|$.

The assertion of the following lemma is reminiscent of the mixing conditions considered in [5].

LEMMA 1. *If $f \in C(T)$ and $g \in L^1(G)$, then*

$$\lim_{\chi \to \infty} \int f(\chi(x))g(x)\,dx = \int f \int g.$$

Proof. For a fixed g, the integral on the left can be viewed as a bounded linear functional F_χ on $C(T)$. Since $\|F_\chi\| \leqslant \|g\|_1$, it suffices to prove

$$\lim_{\chi \to \infty} F_\chi(f) = \int f \int g$$

for the linearly dense set of characters $f(z) = z^m$, $m \in \mathbf{Z}$. If $m = 0$, then clearly

$$F_\chi(f) = \int g = \int f \int g.$$

If $m \neq 0$, then

$$F_\chi(f) = \hat{g}(\chi^{-m}),$$

where $\hat{g} \in C_0(G)$ is the Fourier transform of g. Since

$$\lim_{\chi \to \infty} \chi^{-m} = \infty$$

by (*), we obtain

$$\lim_{\chi \to \infty} F_\chi(f) = 0 = \int f \int g$$

as required.

The following lemma can be viewed as an extension of the Hardy–Littlewood theorem:

LEMMA 2. *Let U be an open neighborhood in T. The set*

$$F(U) = \{x \in G\colon (\forall \chi \in \Phi)\, \chi(x) \notin U\}$$

is nowhere dense and of Haar measure zero.

Proof. Clearly, $F(U)$ is closed. Suppose $V \subset F(U)$, $0 < |V| < \infty$. We let $g = 1_V$, the indicator of V, and $0 \leqslant f \leqslant 1_U$, $0 \neq f \in C(T)$. Then

$$f(\chi(x))g(x) \equiv 0$$

for all $\chi \in \Phi$. On the other hand,

$$\lim_{\chi \to \infty} \int f(\chi(x))g(x)\,dx = |V| \int f > 0$$

by Lemma 1. Since we may choose $\chi \in \Phi$, $\chi \to \infty$, this is a contradiction.

LEMMA 3. *Let $n \geqslant 1$. The set of all $(x_1, \ldots, x_n) \in G^n$ such that some $f\colon \{x_1, \ldots, x_n\} \to T$ cannot be extended to a character contained in the pointwise closure of Φ is of the first category in G^n.*

Proof. It suffices to prove that for any neighborhoods $U_1, \ldots, U_n$ from a countable basis of T the set

$$F(U_1, \ldots, U_n) = \{(x_1, \ldots, x_n) \in G^n\colon (\forall \chi \in \Phi)(\chi(x_1), \ldots, \chi(x_n)) \notin U_1 \times \ldots \times U_n\}$$

is nowhere dense. Suppose, to the contrary, that there are open sets

$V_1, \ldots, V_n$ in G such that

$$0 < |V_j| < \infty \quad \text{and} \quad V_1 \times \ldots \times V_n \subset F(U_1, \ldots, U_n).$$

We let $g_j = 1_{V_j}$ and $0 \leqslant f_j \leqslant 1_{U_j}$, $0 \neq f_j \in C(T)$, $j = 1, \ldots, n$. Now argue as in Lemma 2. If $\chi \in \Phi$, then

$$\prod_{j=1}^{n} f_j(\chi(x_j)) g_j(x_j) \equiv 0,$$

while

$$\int \ldots \int \prod_{j=1}^{n} f_j(\chi(x_j)) g_j(x_j)\, dx_1 \ldots dx_n = \prod_{j=1}^{n} \int f_j(\chi(x)) g_j(x)\, dx \to \prod_{j=1}^{n} |V_j| \int f_j > 0$$

as $\chi \to \infty$, a contradiction.

Now we are in a position to apply Mycielski's theorem on independent sets in topological relation structures [6]. First define for each $n \geqslant 1$ an n-ary relation R_n on G by letting $(x_1, \ldots, x_n) \in R_n$ iff some function

$$f\colon \{x_1, \ldots, x_n\} \to T$$

does not extend to a character belonging to the pointwise closure of Φ. By Lemma 3, R_n is of the first category in G^n. Moreover, the relational structure $\mathfrak{R} = (G, \{R_1, R_2, \ldots\})$ is closed under identification of variables since

$$(x_1, \ldots, x_i, x_j, x_{i+1}, \ldots, x_n) \in R_{n+1} \quad \text{iff} \quad (x_1, \ldots, x_n) \in R_n$$

for any j ($1 \leqslant j \leqslant n$, $n \geqslant 1$). Besides, G is dense in itself if an unbounded Φ exists. Consequently, $\mathfrak{R}$ satisfies the assumptions of Mycielski's theorem [6] asserting the existence of an independent set of cardinality continuum (which can be chosen to be a dense countable union of Cantor sets if G is second countable). Since independence in $\mathfrak{R}$ coincides with our independence defined in Section 2, we have

THEOREM 1. *Let G be an LCA group satisfying* (*). *Then for every unbounded $\Phi \subset \hat{G}$ there exists a Φ-independent set of cardinality continuum. If, in addition, G is second countable, then the independent set can be chosen to be a dense countable union of Cantor sets.*

Remark. If Φ^k, $k = 1, 2, \ldots$, is a sequence of unbounded subsets of $\hat{G}$, then by a slight modification of the above argument we may obtain a set of cardinality continuum which is Φ^k-independent for all $k \geqslant 1$ simultaneously. Indeed, it suffices to consider the relational structure $(G, \{R_n^k\colon n \geqslant 1, k \geqslant 1\})$, where the relations $R_1^k, R_2^k, \ldots$ correspond to the family Φ^k. In particular, there exists a Borel uncountable set $E \subset T$, Φ_p-independent for all $p \geqslant 2$. An analogous remark applies to Theorem 2 below.

Another method of constructing uncountable Φ-independent sets is based on transfinite induction. For the remaining part of the paper, G is a compact abelian group satisfying $(*)$ and Φ is an infinite subset of $\hat{G}$.

First note that if E is Φ-independent, then it is either of measure zero or nonmeasurable. Indeed, the translations Ea, $a \in G$, are almost disjoint by (ii), so the inner measure of E must be zero. Our aim is now to produce nonmeasurable Φ-independent sets. This will be achieved under additional conditions, e.g., G metrizable plus the continuum hypothesis.

Denote by $\mathscr{B}$, $\mathscr{M}$, and $\mathscr{N}$ the Borel σ-algebra, the ideal of first category sets, and the ideal of sets of Haar measure zero in G, respectively. If $\mathscr{I} = \mathscr{M}$ or $\mathscr{I} = \mathscr{N}$, then we consider the cardinal number

$$\operatorname{cov} \mathscr{I} = \min \left\{\operatorname{card} \mathscr{F}\colon\ \mathscr{F} \subset \mathscr{I}, (\exists C \in \mathscr{B} \setminus \mathscr{I})\ C \subset \bigcup_{D \in \mathscr{F}} D\right\}.$$

Clearly, $\aleph_1 \leqslant \operatorname{cov} \mathscr{I} \leqslant \operatorname{card} G$. Also note that if G is metrizable, then by the well-known isomorphism theorems $\operatorname{cov} \mathscr{I}$ is independent of G.

THEOREM 2. *Let G be a compact abelian group satisfying $(*)$ and $\Phi \subset \hat{G}$ be infinite. For any family*

$$\mathscr{C} \subset \mathscr{B} \setminus (\mathscr{M} \cap \mathscr{N}) \quad \text{with } \operatorname{card} \mathscr{C} = \min(\operatorname{cov} \mathscr{M}, \operatorname{cov} \mathscr{N})$$

there exists an uncountable Φ-independent set E such that $E \cap C \neq \emptyset$ for each $C \in \mathscr{C}$.

Proof. Put $\gamma = \min(\operatorname{cov} \mathscr{M}, \operatorname{cov} \mathscr{N})$ and order $\mathscr{C}$ as $\{C_\alpha\colon \alpha < \gamma\}$. We define a transfinite sequence x_α, $\alpha < \gamma$, such that $x_\alpha \in C_\alpha$ and $\{x_\alpha\colon \alpha < \beta\}$ is Φ-independent for each $\beta < \gamma$. To carry out the induction suppose $\{x_\alpha\colon \alpha < \beta\}$ is Φ-independent for some $\beta < \gamma$. We shall find an element $x_\beta \in C_\beta$ such that $\{x_\alpha\colon \alpha \leqslant \beta\}$ is still Φ-independent. First choose any $0 \leqslant \alpha_1 < \ldots < \alpha_n < \beta$ (n finite) and neighborhoods $U_1, \ldots, U_n$ from a countable basis in T. By inductive assumption, the set

$$\Psi = \left\{\chi \in \Phi\colon\ (\chi(x_{\alpha_1}), \ldots, \chi(x_{\alpha_n})) \in U_1 \times \ldots \times U_n\right\}$$

is infinite (we let $\Psi = \Phi$ if $\beta = 0$). By Lemma 2, the set

$$Y(\alpha_1, \ldots, \alpha_n, U_1, \ldots, U_n) = \left\{y \in G\colon\ \{y\} \text{ is } \Psi\text{-independent}\right\}$$

is a dense G_δ of full measure. Since $C_\beta \in \mathscr{B} \setminus \mathscr{M}$ or $C_\beta \in \mathscr{B} \setminus \mathscr{N}$ and there are less than γ sets Y, the intersection

$$C_\beta \cap \bigcap Y(\alpha_1, \ldots, \alpha_n, U_1, \ldots, U_n)$$

is nonempty. Choose any x_β in this intersection. It is clear that for any $0 \leqslant \alpha_1 < \ldots < \alpha_n < \beta$ and neighborhoods $U_1, \ldots, U_n, U_{n+1}$ we have

$$x_\beta \in Y(\alpha_1, \ldots, \alpha_n, U_1, \ldots, U_n),$$

so the set

$$\{\chi\in\Phi\colon (\chi(x_{\alpha_1}), \ldots, \chi(x_{\alpha_n}), \chi(x_\beta))\in U_1\times\ldots\times U_n\times U_{n+1}\}$$

is nonempty. This proves that $x_\beta\notin\{x_\alpha\colon \alpha<\beta\}$ and that $\{x_\alpha\colon \alpha\leqslant\beta\}$ is Φ-independent.

COROLLARY. *If G is compact abelian and metrizable, $\Phi\subset\hat{G}$ is infinite, and* $\operatorname{cov}\mathscr{M}=\operatorname{cov}\mathscr{N}=2^{\aleph_0}$, *then there exists a Φ-independent set E such that $E\cap C\neq\emptyset$ whenever $C\in\mathscr{B}\setminus(\mathscr{M}\cap\mathscr{N})$. In particular, E is nonmeasurable and does not belong to the Baire σ-algebra.*

Note that $\operatorname{cov}\mathscr{M}=\operatorname{cov}\mathscr{N}=2^{\aleph_0}$ is implied by the continuum hypothesis.

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