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$\aleph_1$-*INCOMPACTNESS OF Z*

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In this note we answer a question posed by A. Ehrenfeucht and J. Łoś and stated as P 482 in [3]. Let us say that an algebra $\mathfrak{A}$ is *(equationally) a-incompact*, where a is an infinite cardinal, if there exists a system of a equations, involving constant elements of $\mathfrak{A}$ and unknows, which has no solution in $\mathfrak{A}$, although each subsystem of less than a equations has a solution (1). We shall make a few general observations concerning this notion and then show that the group Z of integers is $\aleph_1$-incompact, thus solving P 482.

A topological space is called *a-compact* iff every open covering has a subcovering of power less than a. We will say that the space is *a-incompact* iff it is not a-compact. Consider the topological power A^a where A, the universe of the algebra $\mathfrak{A}$, is given the discrete topology; if the set of unknowns of a system of equations is taken to be a, then the set of solutions of each given equation forms a closed subset of A^a. It is clear therefore that the a-incompactness of $\mathfrak{A}$ implies that the space A^a is a-incompact.

Now the $\varkappa$-incompactness of the group Z is known for only a few cases: for $\varkappa = \aleph_0$ and (as we shall see) for $\varkappa = \aleph_1$; although it follows from Łoś [1] that when λ is a non-measurable regular cardinal one has $\varkappa$-incompactness of Z for some $\varkappa$ in the interval $\lambda \leqslant \varkappa \leqslant 2^\lambda$. On the other hand, the $\varkappa$-incompactness of the space $N^\varkappa$ ($N = \{0, 1, 2, \ldots\}$; with the discrete topology) is known for every cardinal belonging to the smallest class M containing $\aleph_0$ and (i) containing $\varkappa^+$ if it contains $\varkappa$; (ii) containing $2^\varkappa$ if it contains $\varkappa$; (iii) containing $\sum_{\xi<\lambda} \beta_\xi$ if it contains λ and every cardinal β_ξ.

Thus a number of questions are open, the most general one being: Does $\varkappa$-incompactness of $N^\varkappa$ imply the $\varkappa$-incompactness of Z (**P 739**)? And more specifically: Is Z $\aleph_2$-incompact (**P 740**)?

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(1) This is not the simple negation of the notion called *equationally a-compact* and studied in [3].

Remark. Cardinal numbers in the class M are known to have a possibly stronger property, which implies the other; namely that there exists in $N^{\varkappa}$ a closed, discrete subset of power $\varkappa$. Thus, Mycielski [4] proved that the class of cardinals with this stronger property is closed under operations (i) and (iii) above. Mrówka [2] found that it is closed under (ii), further, that if λ is nonmeasurable then 2^{λ} is in the class.

I will give below a proof for the first mentioned result of Mrówka, since I have not found it in the literature. No proof for his second result is known to me.

THEOREM 1. *Z is $\aleph_1$-incompact.*

Proof. Let ω_1 be the least uncountable ordinal (i.e. $\omega_1 = \aleph_1$), and let P be the set of prime natural numbers. By a familiar result of Sierpiński there exists a system $\{P_\mu | \mu < \omega_1\}$ of infinite subsets of P, such that $P_\mu \cap P_\nu$ is finite whenever $\mu \neq \nu$. Since $\{\mu | \mu < \nu\}$ is countable for each $\nu < \omega_1$ we can easily construct a system $p(\mu, \nu) \epsilon P$ $(\mu < \nu < \omega_1)$ so that: (1) $p(\mu, \nu) \epsilon P_\nu - P_\mu$; and (2) $p(\mu_0, \nu) \neq p(\mu_1, \nu)$ whenever $\mu_0 < \mu_1 < \nu$.

Now consider the equations:

$$\Sigma: x_\eta - x_\zeta = 1 + p(\zeta, \eta) \cdot y_{\zeta,\eta} \qquad (\zeta < \eta < \omega_1);$$

where x_η, x_ζ and $y_{\zeta,\eta}$ are the unknowns. Such an equation for integers x_η, x_ζ and $y_{\zeta,\eta}$ implies $x_\eta \neq x_\zeta$. Hence the entire set of equations has no solution. However, we can show that, for each $\gamma < \omega_1$, there is a system of integers $\{a_\eta | \eta < \gamma\}$ such that $a_\eta \equiv a_\zeta + 1 (\mathrm{mod}\, p(\zeta, \eta))$ whenever $\zeta < \eta < \gamma$; so every countable subset of Σ is solvable.

There is no loss of generality in assuming $\omega \leqslant \gamma$. Order $\{\xi | \xi < \gamma\}$ in a sequence: $\xi_0, \xi_1, \ldots, \xi_n, \ldots$ $(n < \omega)$ without any repetitions. Letting $a_{\xi_0} = b_0 = 0$, we try to find $b_1 (= a_{\xi_1}), b_2, \ldots \epsilon Z$ so that for $l < k < \omega$:

$A_{l,k}$. If $\xi_l < \xi_k$, then $b_k \equiv b_l + 1 (\mathrm{mod}\, p(\xi_l, \xi_k))$.

$B_{l,k}$. If $\xi_k < \xi_l$, then $b_k \equiv b_l - 1 (\mathrm{mod}\, p(\xi_k, \xi_l))$.

$C_{l,k}$. If $p = p(\xi_m, \xi_k) = p(\xi_m, \xi_l)$ for some $m > k, l$, then $b_k \equiv b_l\ (\mathrm{mod}\, p)$.

Suppose that $b_0, \ldots, b_{n-1}$ have been found so that A, B, C are true whenever $l < k < n$. Then we have to solve the system of congruences $(A_{k,n}, B_{k,n}, C_{k,n}\ (k < n))$ for b_n. Notice that there are only finitely many congruences altogether in this system, since by assumptions (1), (2) concerning the $p(\ ,\ )$, all the moduli p occurring in $C_{k,n}$ lie in the finite set $P_{\xi_k} \cap \cap P_{\xi_n}$. Hence (by the Chinese Remainder Theorem) b_n can be found if, whenever the same prime p occurs as the modulus in two of the new congruences, then the right sides of these congruences are congruent modulo p.

To check that this is so, observe that for $k < n$, $A_{k,n}$ does not share the modulus with $A_{l,n}$ $(l \neq k)$ nor with any of the other congruences —

because of conditions (1) and (2). For the same reason there is no sharing between any $B_{k,n}$ and $C_{l,n}$. (If $p = p(\xi_m, \xi_n)$, then $p \in P_{\xi_n} - P_{\xi_m}$, but $p(\xi_n, \xi_k) \in P_{\xi_k} - P_{\xi_n}$.) Also, if in $B_{k,n}, B_{l,n}$ we have $p(\xi_n, \xi_k) = p(\xi_n, \xi_l)$ where, say, $l < k < n$ then the condition $C_{l,k}$, which is true by induction assumption, gives $b_k - 1 \equiv b_l - 1 (\bmod p(\xi_n, \xi_k))$. Finally, if two of the new congruences given by C share a modulus – say $p = p(\xi_r, \xi_n) = p(\xi_r, \xi_k)$, $q = p(\xi_s, \xi_n) = p(\xi_s, \xi_l)$, where $l < k < n < r, s$ and $p = q$ – then $r = s$ by (2) and so $b_k \equiv b_l (\bmod p)$, by $C_{l,k}$ (the induction assumption).

Thus a suitable value of b_n can be found and the construction continued indefinitely. Setting $a_\eta = b_m$ where $\eta = \xi_m$, we have the desired result.

THEOREM 2. *Let $\lambda = 2^\varkappa$. If $N^\varkappa$ has a closed, discrete subset of power $\varkappa$ then N^λ has a closed, discrete subset of power λ.*

Proof. Let $\mathscr{T}$ be such a set for $N^\varkappa$. Select one member of $\mathscr{T}$ and enumerate the rest, so that

$$\mathscr{T} = \{\theta\} \cup \{f_\beta | \beta < \varkappa\},$$

and $\theta, f_0, \ldots, f_\beta, \ldots$ are distinct. Let $S = \varkappa \cup (N^\varkappa \times \varkappa)$; since N^S and N^λ are homeomorphic, we only have to define a subset of N^S. For each $G \in N^S$ we put

$$G_*(\beta) = G(\beta) \quad (\beta \in \varkappa),$$

$$G_h(\beta) = G(\langle h, \beta \rangle) \quad (h \in N^\varkappa, \beta \in \varkappa).$$

A member $G \in N^S$ is completely specified by the functions $G_*, G_h \in N^\varkappa$ ($h \in N^\varkappa$). For each $g \in N^\varkappa$, we now define a function $H^{(g)} \in N^S$ so that

$$H^{(g)}_* = g; \quad H^{(g)}_g = \theta; \quad H^{(g)}_h = f_\beta;$$

where $g(\beta) \neq h(\beta) \wedge (\forall \gamma < \beta) g(\gamma) = h(\gamma)$.

We define $\mathscr{S} = \{H^{(g)} | g \in N^\varkappa\}$, and clearly $\mathscr{S}$ has the potency λ. To see that $\mathscr{S}$ is closed and discrete, we use the fact that the projections $G \to G_*, G \to G_h$, for fixed h, are continuous mappings from N^S to $N^\varkappa$. The following are therefore closed sets in N^S (for $h \in N^\varkappa, \beta \in \varkappa$):

$$\mathscr{S}_h = \{G | G_h \in \mathscr{T} - \{\theta\} \vee G = H^{(h)}\};$$

$$\mathscr{S}_{h,\beta} = \{G | G_h \in \mathscr{T} - \{f_\beta\} \vee G_*(\beta) \neq h(\beta)\};$$

because $\mathscr{T} - \{\theta\}, \mathscr{T} - \{f_\beta\}$ are closed. Now $\mathscr{S}$ is identical with the intersection of all of these sets: in fact if $G \in N^S$ and, say, $G_* = h$, then $G \in \mathscr{S}_{h,\beta}$ for all β implies $G_h = \theta$; and then $G \in \mathscr{S}_h$ implies $G = H^{(h)}$. Thus $\mathscr{S}$ is closed. Moreover, for any h, $\{G | G_h \notin T - \{\theta\}\}$ is a neighbourhood of $H^{(h)}$ which contains no other member of $\mathscr{S}$. Hence $\mathscr{S}$ is discrete.

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