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TRIGONOMETRIC INTERPOLATION, I

BY

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1. Preliminaries. Throughout this paper the function $f(s)$ is 2π-periodic, Riemann-integrable over the interval $\langle -\pi, \pi\rangle$, and is a subject to further restrictions specified below.

Consider the n-th interpolating polynomial

$$\frac{a_0^{(n)}}{2} + \sum_{k=1}^{n} (a_k^{(n)} \cos ks + b_k^{(n)} \sin ks)$$

which coincides at the points

$$s_l = s_l^{(n)} = 2\pi l/(2n+1) \qquad (l = 0, \pm 1, \pm 2, \ldots) \tag{1}$$

with $f(s_l)$. Write

$$I_\nu^{(n)}(x;f) = \frac{a_0^{(n)}}{2} + \sum_{k=1}^{\nu} (a_k^{(n)} \cos kx + b_k^{(n)} \sin kx) \qquad (0 \leqslant \nu \leqslant n).$$

Since

$$a_k^{(n)} = \frac{2}{2n+1} \sum_{l=-n}^{n} f(s_l) \cos ks_l, \qquad b_k^{(n)} = \frac{2}{2n+1} \sum_{l=-n}^{n} f(s_l) \sin ks_l, \tag{2}$$

we have

$$I_\nu^{(n)}(x;f) = \frac{2}{2n+1} \sum_{l=-n}^{n} f(s_l) D_\nu(s_l - x), \tag{3}$$

where

$$D_\nu(z) = \frac{1}{2} + \sum_{k=1}^{\nu} \cos kz = \frac{\sin(\nu+\frac{1}{2})z}{2\sin\frac{1}{2}z}.$$

Introduce a convenient integral notation analogous to that of [3], p. 4. Let $\omega_n(s)$ be the step function which is equal to $2\pi l/(2n+1)$ for

$s \in \langle s_{l-1}, s_l)$ $(l = 0, \pm 1, \pm 2, \ldots)$. Consider an interval $\langle a, b\rangle$; suppose that $s_{a-1} < a \leqslant s_a < s_{a+1} < \ldots < s_\beta < b \leqslant s_{\beta+1}$. Then we shall write

$$\int_a^b \varphi(s)\, d\omega_n(s) = \frac{2\pi}{2n+1} \sum_{l=a}^{\beta} \varphi(s_l) \tag{4}$$

for any function $\varphi(s)$ defined in $\langle a, b\rangle$. If φ is continuous in this interval, integral (4) exists in the Riemann-Stieltjes sense. If φ is 2π-periodic, then $\int_a^{a+2\pi} \varphi(s)\, d\omega_n(s)$ is independent of a. In particular, by the above convention,

$$a_k^{(n)} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \cos ks\, d\omega_n(s), \qquad b_k^{(n)} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \sin ks\, d\omega_n(s)$$

and

$$I_\nu^{(n)}(x; f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) D_\nu(s-x)\, d\omega_n(s) = \frac{1}{\pi} \int_0^{2\pi} f(s) D_\nu(s-x)\, d\omega_n(s).$$

Let $g(s)$ be a function defined in the interval $I = \langle a, b\rangle$. Denote by Π an arbitrary partition of I, generated by the points

$$a = x_0 < x_1 < x_2 < \ldots < x_{m-1} < x_m = b.$$

Write, for a given $p > 0$,

$$V_p(g; I) = \sup_{\Pi} \Big\{ \sum_{i=1}^{m} |g(x_i) - g(x_{i-1})|^p \Big\}^{1/p}.$$

As it is well-known, the quantity $V_p(g; I)$ is called the *p-th variation of* f; if $V_p(g; I) < \infty$, we say that g *is of bounded p-th variation over* I.

In this Note we investigate the convergence of trigonometric polynomials (3) and their coefficients (2) for some functions f.

2. Fundamental lemmas. Start with the following

2.1. LEMMA. *Let* $S = (s_{l_0}, s_{l_1}, \ldots, s_{l_r})$ *be an arbitrary subsequence of the nodes* $s_{-n}, s_{-n+1}, \ldots, s_{n+1}$ *defined by* (1), *and let* $q > 1$. *Then*

$$\max_S \Big\{ \sum_{i=1}^{r} \Big| \int_{s_{l_{i-1}}}^{s_{l_i}} D_\nu(s-x)\, d\omega_n(s) \Big|^q \Big\}^{1/q}$$

is uniformly bounded in ν, n, x $(0 \leqslant \nu \leqslant n,\ -\pi \leqslant x \leqslant \pi)$.

Proof. The points $s_{l_0}, s_{l_1}, \ldots, s_{l_{r-1}}$ belong to the interval $\langle x-2\pi, x+2\pi\rangle$ whenever $x \in \langle -\pi, \pi\rangle$. Set

$$a_m = x + \frac{2\pi m}{2\nu+1} \qquad (m = 0, \pm 1, \pm 2, \ldots, \pm(2\nu+1)).$$

Consider any interval $\langle a, b\rangle \subset \langle a_m, a_p\rangle$. Applying the inequality

$$(5) \qquad \left|\sum_{i=a}^{\beta} \sin(\nu+\tfrac{1}{2})(s_i-x)\right| \leqslant \frac{2n+1}{2\nu+1} \qquad (-\pi \leqslant x \leqslant \pi),$$

valid for arbitrary integers α, β, we easily prove that

$$(6) \qquad \left|\int_a^b D_\nu(s-x)\,d\omega_n(s)\right|$$

$$\leqslant 5\pi \max\left(\frac{1}{|m|+1}, \frac{1}{|p|+1}, \frac{1}{2\nu+2-|m|}, \frac{1}{2\nu+2-|p|}\right)$$

if $-2\nu-1 \leqslant m < p \leqslant 0$ or $0 \leqslant m < p \leqslant 2\nu+1$, $0 \leqslant \nu \leqslant n$. In the case of $\langle a, b\rangle = \langle a_m, a_{m+1}\rangle$, the left-hand side of (6) can be replaced by

$$\int_{a_m}^{a_{m+1}} |D_\nu(s-x)|\,d\omega_n(s).$$

Let us write

$$\sum_{i=1}^{r}\left|\int_{s_{l_{i-1}}}^{s_{l_i}} D_\nu(s-x)\,d\omega_n(s)\right|^q = \Big(\sum_i{}' + \sum_i{}''\Big)\left|\int_{s_{l_{i-1}}}^{s_{l_i}} D_\nu(s-x)\,d\omega_n(s)\right|^q,$$

where $\sum'$ denotes summation over all those i for which $\langle s_{l_{i-1}}, s_{l_i}\rangle$ contains no a_m $(m = 0, \pm 1, \ldots, \pm(2\nu+1))$.

Obviously

$$\sum_i{}'\left|\int_{s_{l_{i-1}}}^{s_{l_i}} D_\nu(s-x)\,d\omega_n(s)\right|^q \leqslant \sum_{j=-2\nu}^{2\nu+1}\left\{\int_{a_{j-1}}^{a_j} |D_\nu(s-x)|\,d\omega_n(s)\right\}^q$$

and, by (6), the last sums are uniformly bounded. Also, applying (6), we observe that

$$\sum_i{}''\left|\int_{s_{l_{i-1}}}^{s_{l_i}} D_\nu(s-x)\,d\omega_n(s)\right|^q \leqslant C,$$

where C is an absolute constant. Hence the assertion follows (cf. [2], p. 272-274).

Now, we shall prove a similar

2.2. LEMMA. *Under the assumption of* 2.1,

$$\max_S\left\{\sum_{i=1}^{r}\left|\int_{s_{l_{i-1}}}^{s_{l_i}} {\cos \atop \sin} ks\,d\omega_n(s)\right|^q\right\}^{1/q} \leqslant \frac{5^{1/q}\pi}{k^{1-1/q}} \qquad (1 \leqslant k \leqslant n).$$

Proof. Consider the cosine case only. Let us put

$$b_m = \frac{2m-1}{2k}\pi \quad \text{for } m = 0, \pm 1, \pm 2, \ldots, \pm(k-1),$$

$$b_{-k} = -\pi, \quad b_k = \frac{2k-1}{2k}\pi, \quad b_{k+1} = \pi,$$

and write

$$\sum_{i=1}^{r} \Big| \int_{s_{l_{i-1}}}^{s_{l_i}} \cos ks\, d\omega_n(s) \Big|^q = \Big(\sum_i{}' + \sum_i{}''\Big) \Big| \int_{s_{l_{i-1}}}^{s_{l_i}} \cos ks\, d\omega_n(s) \Big|^q;$$

where $\sum'$ is extended over all those i for which $\langle s_{l_{i-1}}, s_{l_i}\rangle$ contains no b_m.

Observing that, for any $\langle a, b\rangle \subset \langle b_m, b_p\rangle$,

$$\Big| \int_a^b \cos ks\, d\omega_n(s) \Big| \leqslant \pi/k \quad \text{when } -k \leqslant m < p \leqslant k+1,$$

$$\int_{b_m}^{b_{m+1}} |\cos ks|\, d\omega_n(s) \leqslant \pi/k \quad \text{when } -k+1 \leqslant m \leqslant k-1,$$

$$\Big\{ \int_{b_{-k}}^{b_{-k+1}} |\cos ks|\, d\omega_n(s) \Big\}^q + \Big\{ \int_{b_k}^{b_{k+1}} |\cos ks|\, d\omega_n(s) \Big\}^q \leqslant (\pi/k)^q,$$

and reasoning as before, we get the estimate as desired (cf. [2], p. 275-276).

Finally, an analogue of the Riemann-Lebesgue theorem will be given.

2.3. Lemma. *Let $0 < \delta < \pi$. Then*

(i) $$\lim_{\nu\to\infty} \int_{-\pi}^{x-\delta} f(s) D_\nu(s-x)\, d\omega_n(s) = 0, \quad \lim_{\nu\to\infty} \int_{x+\delta}^{\pi} f(s) D_\nu(s-x)\, d\omega_n(s) = 0$$

uniformly in $x \in \langle -\pi+\delta, \pi-\delta\rangle$, and

(ii) $$\lim_{\nu\to\infty} \int_{0}^{x-\delta} f(s) D_\nu(s-x)\, d\omega_n(s) = 0, \quad \lim_{\nu\to\infty} \int_{x+\delta}^{2\pi} f(s) D_\nu(s-x)\, d\omega_n(s) = 0$$

uniformly in $x \in \langle \delta, 2\pi-\delta\rangle$.

Proof of (i). Consider only the integral

$$J_{\nu,n}(x) = \int_{x+\delta}^{\pi} f(s) D_\nu(s-x)\, d\omega_n(s) \quad (-\pi+\delta \leqslant x \leqslant \pi-\delta).$$

Put

$$F_x(s) = \frac{f(s)}{2\sin\frac{1}{2}(s-x)}, \quad M = \sup_{-\pi\leqslant s\leqslant\pi} |f(s)|.$$

Given a positive $\lambda < \delta/2$, there is a partition

$$-\pi = z_1 < z_2 < \ldots < z_k < \ldots < z_{m+1} = \pi$$

such that

$$\max_{1\leqslant k\leqslant m} |z_{k+1}-z_k| < \lambda \quad \text{and} \quad \sum_{k=1}^{m} (z_{k+1}-z_k) \operatorname*{Osc}_{z_k\leqslant s\leqslant z_{k+1}} f(s) < \lambda.$$

Hence, if $z_\varrho < x+\delta \leqslant z_{\varrho+1}$, we have

$$\sum_{k=\varrho}^{m} (z_{k+1}-z_k) \operatorname*{Osc}_{z_k\leqslant s\leqslant z_{k+1}} F_x(s) < \frac{\lambda}{2\sin\frac{1}{4}\delta} + \frac{\pi\lambda M}{2\sin^2\frac{1}{4}\delta}. \tag{7}$$

Let us write

$$J_{\nu,n}(x) = \Big(\int_{x+\delta}^{z_{\varrho+1}} + \int_{z_{\varrho+1}}^{\pi}\Big) F_x(s)\sin(\nu+\tfrac{1}{2})(s-x)\,d\omega_n(s) = J'+J''.$$

Evidently,

$$|J'| \leqslant \int_{z_\varrho}^{z_{\varrho+1}} |F_x(s)|\,d\omega_n(s) \leqslant \frac{M}{2\sin\frac{1}{4}\delta}\Big(\lambda + \frac{2\pi}{2n+1}\Big),$$

and

$$|J''| \leqslant \sum_{k=\varrho+1}^{m} \int_{z_k}^{z_{k+1}} |F_x(s)-F_x(z_k)|\,d\omega_n(s) +$$
$$+ \sum_{k=\varrho+1}^{m} |F_x(z_k)| \Big|\int_{z_k}^{z_{k+1}} \sin(\nu+\tfrac{1}{2})(s-x)\,d\omega_n(s)\Big|$$
$$\leqslant \frac{\lambda}{2\sin\frac{1}{4}\delta} + \frac{\pi\lambda M}{2\sin^2\frac{1}{4}\delta} + \frac{2\pi m M}{(2n+1)\sin\frac{1}{4}\delta} + \frac{\pi m M}{(2\nu+1)\sin\frac{1}{4}\delta}$$

by (7) and (5).

Thus

$$|J_{\nu,n}(x)| \leqslant \frac{\lambda(M+1)}{\sin\frac{1}{4}\delta} + \frac{\pi\lambda M}{\sin^2\frac{1}{4}\delta} \qquad (-\pi+\delta \leqslant x \leqslant \pi-\delta)$$

for ν and n large enough, which completes the proof (cf. [1], p. 461; [3], p. 17).

3. Main results. First we shall present an analogue of the well-known Young's test.

3.1. THEOREM. *Suppose that $f(s)$ is of bounded p-th variation over an interval $\langle A, B\rangle$, $p \geqslant 1$. Then*

(i) *we have*

$$\lim_{\nu\to\infty} I_\nu^{(n)}(x;f) = f(x) \tag{8}$$

at every point of continuity of f in (A, B);

(ii) *if f is continuous at every point x of a closed interval $\langle a, b\rangle \subset (A, B)$, the convergence* (8) *is uniform in $\langle a, b\rangle$.*

Proof of (ii). Consider the case $-\pi \leqslant A < B \leqslant \pi$. Suppose that $p_1 > p$. Choose, for an arbitrary $\varepsilon > 0$, a positive $\delta \leqslant \min(a-A, B-b)$ such that

$$|f(x+h)-f(x)| < \varepsilon \quad \text{and} \quad V_{p_1}(f; \langle x-\delta, x+\delta\rangle) < \varepsilon,$$

when $x \in \langle a, b\rangle$ and $|h| < \delta$ (see (8.2a) of [2]). Write

$$I_\nu^{(n)}(x; f) - f(x) = \frac{1}{\pi}\Big(\int_{-\pi}^{x-\delta} + \int_{x-\delta}^{x} + \int_{x}^{x+\delta} + \int_{x+\delta}^{\pi}\Big)\{f(s)-f(x)\}D_\nu(s-x)\,d\omega_n(s)$$
$$= \frac{1}{\pi}(J_1+J_2+J_3+J_4).$$

By 2.3, the integrals J_1, J_4 tend to zero as $\nu \to \infty$, uniformly in $\langle a, b\rangle$. The Abel transformation leads to

$$J_3 = \frac{2\pi}{2n+1}\sum_{j=k}^{m-1}\sum_{l=k}^{j}\{f(s_j)-f(s_{j+1})\}D_\nu(s_l-x) +$$
$$+ \frac{2\pi}{2n+1}\{f(s_m)-f(x)\}\sum_{l=k}^{m} D_\nu(s_l-x) = J_3' + J_3'',$$

where the nodes $s_k, s_{k+1}, \ldots, s_m$ are in $\langle x, x+\delta)$. Applying inequality (5.1) of [2] and our Lemma 2.1, we obtain

$$|J_3'| \leqslant C(p_1)\, V_{p_1}(f; \langle x, x+\delta\rangle) \quad (x \in \langle a, b\rangle,\ 1 \leqslant \nu \leqslant n),$$

with a constant $C(p_1)$ depending only on p_1. Also, by 2.1, there is an absolute constant K such that

$$\frac{2\pi}{2n+1}\Big|\sum_{l=k}^{m} D_\nu(s_l-x)\Big| \leqslant K.$$

Consequently,

$$|J_3| < \{C(p_1)+K\}\varepsilon \quad (x \in \langle a, b\rangle,\ 1 \leqslant \nu \leqslant n),$$

and J_3 can be replaced by J_2, too. Thus the result follows.

The proof of (i) runs on the same line (cf. [2], p. 274 and [3], p. 17).

By the analysis of the proof of (5.5) in [3], pp. 17-18, we get

3.2. THEOREM. *Let $f(s)$ be continuous at every point of a closed interval $\langle a, b\rangle$. Suppose that, for every $x \in \langle a, b\rangle$, there is a function $\mu_x(s)$ non-decreasing in an interval $\langle 0, \eta\rangle$ such that*

$$|f(x \pm s) - f(x)| \leqslant \mu_x(s) \quad (0 \leqslant s \leqslant \eta)$$

and

$$\lim_{\sigma\to 0+}\int_0^\sigma \frac{\mu_x(s)}{s}\,ds = 0$$

uniformly in $x\in\langle a,b\rangle$. *Then the convergence* (8) *is uniform in* $\langle a,b\rangle$.

Now, three estimates for Fourier-Lagrange coefficients (2) will be given.

3.3. THEOREM. *Suppose that* $f(s)$ *is of bounded p-th variation over* $\langle -\pi,\pi\rangle$, *with* $p>1$. *Set* $V = V_p(f;\langle -\pi,\pi\rangle)$.

(i) *If* $1\leqslant k\leqslant n$ $(n = 1,2,\ldots)$, *then*

$$|a_k^{(n)}|\leqslant \frac{C_1 V}{k^{1/p'}}\quad \textit{for any } p'>p \tag{9}$$

and

$$|a_k^{(n)}|\leqslant \frac{C_2 V}{k^{1/p}}\log(n+1), \tag{10}$$

where C_1 *and* C_2 *are some positive constants depending on* p *and* p' *only.*

(ii) *If* $0<c\leqslant k/n\leqslant 1$, $c = \text{const}$, *we have*

$$|a_k^{(n)}|\leqslant \left(\frac{2}{c}\right)^{1-1/p}\frac{V}{k^{1/p}}. \tag{11}$$

Estimates (9)-(11) *remain true for* $b_k^{(n)}$, *too.*

Proof. By the Abel transformation,

$$a_k^{(n)} = \frac{2}{2n+1}\sum_{j=-n}^{n-1}\sum_{l=-n}^{j}\{f(s_j)-f(s_{j+1})\}\cos ks_l.$$

(i) Choose a number $q>1$ such that $1/p+1/q>1$. Then inequality (5.1) of [2] and Lemma 2.2 give

$$|a_k^{(n)}|\leqslant \frac{5^{1/q}V}{k^{1-1/q}}\left\{1+\zeta\left(\frac{1}{p}+\frac{1}{q}\right)\right\}.$$

By putting $p' = q/(q-1)$, we get (9).

Choose, next, a $q>1$ such that $1/p+1/q = 1$. In this case inequality (5.1) of [2] should be replaced by

$$\left|\sum_{j=1}^{n}\sum_{l=1}^{j}a_l b_j\right|\leqslant (2+\log n)S_{p,q}(a,b).$$

Applying it together with 2.2, we conclude (10).

(ii) In view of Hölder's inequality (see also [3], p. 15-16)

$$|a_k^{(n)}| \leqslant \frac{1}{k}\left\{\sum_{j=-n}^{n-1} |f(s_j)-f(s_{j+1})|^p\right\}^{1/p}(2n)^{1-1/p} \leqslant \frac{V}{k}(2n)^{1-1/p},$$

and (11) follows.

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