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IDEAL AND REPRESENTATION THEORY OF THE L^1-ALGEBRA OF A GROUP WITH POLYNOMIAL GROWTH

BY

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1. Introduction. In this paper* we study the ideal and representation theory of the algebra $L^1(G)$, where G is a locally compact group which is of polynomial growth (notation: $G \in$ [PG]). We state the definition of the class [PG] in Section 2. For further information concerning this important class of groups we refer the reader to Palmer's useful article [18], and to the extensive bibliography of this article. In many cases our arguments are presented in a more general Banach *-algebra setting. We consider questions concerning symmetry, ideal theory, and representation theory of Banach *-algebras that have basic properties in common with $L^1(G)$, $G \in$ [PG]. The applications to $L^1(G)$ are immediate.

We mention briefly the most interesting of these applications (in each case $G \in$ [PG]). Concerning symmetry, it is shown in Section 3 that if G has a compact subgroup K with the property that every algebraically irreducible Banach-representation of $L^1(G)$ is K-finite, then $L^1(G)$ is symmetric. This result is more general than Gangolli's theorem [7] that $L^1(G)$ is symmetric if G is a Euclidean motion group (our methods are entirely different from Gangolli's). Let γ denote the largest C^*-norm on $L^1(G)$ and let $C^*(G)$ be the completion of $L^1(G)$ in the norm γ. In Section 4 we show that there is a close relation between the closed ideals of $C^*(G)$ and the γ-closed ideals of $L^1(G)$. Explicitly, if I is a closed ideal of $C^*(G)$, then $I \cap L^1(G)$ is dense in I in $C^*(G)$ (Theorem 4.2). This central result leads to a structure theorem for $L^1(G)$ in the case where $C^*(G)$ has Hausdorff structure space (Theorem 4.3) and to information about certain types of closed ideals of $L^1(G)$.

In Section 5 the results of Section 4 are applied to the representation theory of $L^1(G)$. It is shown that if G is GCR (postliminaire) and symmetric, then every continuous irreducible Banach-representation of $L^1(G)$ with γ-closed kernel is finite dimensionally spanned and Naĭmark-related to an irreducible *-representation of $L^1(G)$ (Theorem 5.1). Lastly, we prove

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that every continuous irreducible representation of a Moore group (see [18]) on a normed linear space is finite dimensional. This extends to a general setting results of Godement [9], p. 136-137, and Kaplansky [13], Theorem 3.

Since this paper was written, the interesting paper [2] has appeared with which there is some overlap.

2. Notation and preliminaries. Let A be a Banach algebra with norm $\|\cdot\|_A$. For $f \in A$, $\langle f \rangle$ is the closed subalgebra of A generated by f. We denote the spectrum of f in A by $\mathrm{sp}(f; A)$, and the spectral radius by $\varrho_A(f)$. When A has an involution *, we let A_{sa} be the set of all self-adjoint elements f in A (i.e. $f = f^*$). In general, we will assume that A has an involution and that there exists a C^*-norm on A. Following Rickart in [20], p. 181, we call a Banach *-algebra with a C^*-norm an *A^*-algebra*. An A^*-algebra always has a largest C^*-norm which we denote by γ_A. We often drop the subscript A in the notation given above when the algebra A is clearly understood from the context. We use the notation $\bar{A}$ for the completion of A with respect to the norm γ. If $E \subset A$, then $\bar{E}$ denotes the closure of E in $\bar{A}$.

Let X be a normed linear space. We usually denote the given norm on X by $\|\cdot\|_X$. We let $B(X)$ be the algebra of all bounded linear operators on X and let $F(X)$ be the set of all $T \in B(X)$ such that T has finite-dimensional range.

A representation π of A on a normed linear space X is an algebra homomorphism of A into $B(X)$. The representation π is *irreducible* if π is nonzero and the only closed π-invariant subspaces of X are $\{0\}$ and X. We call a nonzero representation π *algebraically irreducible* if $\{0\}$ and X are the only π-invariant subspaces of X. We often use the pair (π, X) to indicate a given representation π with representation space X.

Let I be an ideal (considered as two-sided ideal) of A. Then I has no ideal divisors in A if whenever J and K are ideals of A with $JK \subset I$, then either $J \subset I$ or $K \subset I$. If π is an irreducible representation of A, then, by [13], Lemma 2.5, $\ker(\pi)$ (the kernel of π) is an ideal with no ideal divisors in A.

For an ideal I of A, let A/I be the usual quotient algebra of A mod I. For $f \in A$, we denote the residue class of A/I that contains f by $f+I$.

The notations $\mathrm{PRIM}(A)$, $\mathrm{PRIM}^*(A)$, $\mathrm{PRIM}(G)$, $C^*(G)$, and [HER] are used as in [18]; here G is a locally compact (LC) group.

Let G be an LC group. If B is a measurable subset of G, then $|B|$ is the Haar measure of B. The group G is of *polynomial growth* (notation: $G \in$ [PG]) if for every compact subset K of G there exists a positive integer m such that

$$|K^n| = O(n^m) \quad \text{as } n \to +\infty.$$

For a particular function f on G, $\mathrm{supp}(f)$ is the smallest closed subset E of G such that f vanishes outside of E. When $f, g \in L^1(G)$, we use $f*g$ for the convolution of f with g.

An algebra norm $\|\cdot\|$ on an algebra B is a *Q-norm* provided that the set of quasi-regular elements (or invertible elements if B has an identity) is open in the $\|\cdot\|$-topology on B. We need the following facts concerning Q-norms:

(2.1) *Let $\|\cdot\|$ be an algebra norm on an algebra B. Then the following are equivalent*:

(1) $\|\cdot\|$ *is a Q-norm on* B;

(2) $\varrho_B(f) \leqslant \|f\|$ $(f \in B)$;

(3) $\varrho_B(f) = \lim\limits_{n\to\infty}(\|f^n\|^{1/n})$ $(f \in B)$.

For the proof see [22], Lemma 2.1.

(2.2) *Let A be an A^*-algebra. A is symmetric if and only if γ_A is a Q-norm on A.*

Proof. A is symmetric means that $\mathrm{sp}(f^*f; A)$ is nonnegative for all $f \in A$. By [3], Theorem 5, p. 226, this is equivalent to A being hermitian, i.e. $\mathrm{sp}(f; A)$ being real for all $f \in A_{sa}$. Statement (2.2) follows from [17], Theorem, p. 523.

(2.3) *If B is a Banach algebra, A is a subalgebra of B, and $\|\cdot\|_B$ is a Q-norm on A, then for $f \in A$*

$$\mathrm{bndry}(\mathrm{sp}(f; A)) \subset \mathrm{bndry}(\mathrm{sp}(f; B)).$$

Proof. The standard proof for the special case of this result where A is a closed subalgebra of B works in this more general setting (see [20], Theorem (1.6.12)).

(2.4) *If $\|\cdot\|$ is a Q-norm on an algebra B and I is a $\|\cdot\|$-closed ideal of B, then $\|f+I\|_q = \inf\{\|f-g\|\colon g \in I\}$ is a Q-norm on A/I.*

(2.5) *If A is a (Jacobson) semisimple Banach algebra with dense socle, then every algebra norm on A is a Q-norm.*

Proof. A Banach algebra with dense socle is a modular annihilator algebra by [23], Lemma 3.11. Then (2.5) follows from [22], p. 375-376 (note especially Lemma 2.8).

3. The question of symmetry. In the case where $G \in [\mathrm{PG}]$ and G is compactly generated, it was shown by Pytlik in [19] that $L^1(G)$ contains a dense $*$-subalgebra which is a symmetric Banach algebra. Also, $L^1(G)$ has the same property when G is locally finite as T. Pytlik and J. Jenkins have recently shown independently. It is natural to ask how this information affects the question of the symmetry of $L^1(G)$. That $L^1(G)$ need not be symmetric even though G is locally finite was shown by a recent example in [7], Section 6. We state a general question.

QUESTION. Let A be a Banach $*$-algebra and let B be a symmetric Banach $*$-algebra continuously and densely $*$-embedded in A. Under what conditions is A symmetric?

Certainly, given the hypotheses stated in the question, the assumption that A is commutative suffices to imply that A is symmetric (in this case it is enough to verify that, for every multiplicative linear functional φ on A, $\varphi(f)$ is real for all $f \in A_{sa}$). Also, it is not difficult to see that if B is a dense symmetric $*$-ideal in A, then A is symmetric. A number of useful criteria for symmetry of a Banach $*$-algebra are contained in Leptin's paper [14]. In particular, the preceding assertion is contained in [14], Satz 1. We prove a result relevant to the question above which has application to $L^1(G)$ for certain $G \in [PG]$.

THEOREM 3.1. *Let A be a Banach $*$-algebra and assume that B is a dense symmetric $*$-subalgebra of A. Assume that A has the property that if (π, X) is any algebraically irreducible representation of A, then $\pi(A) \cap F(X)$ is dense in $\pi(A)$ with respect to the natural quotient norm on $A/\ker(\pi) \approx \pi(A)$. Then A is symmetric.*

Proof. Let (π, X) be an arbitrary algebraically irreducible representation of A. If $f = f^* \in B$, then by hypothesis $\mathrm{sp}(f; B)$ is real. It follows that $\mathrm{sp}(f; A)$ is real. Since $\mathrm{sp}(f; A) \supset \mathrm{sp}(\pi(f); \pi(A))$, we have

$(*)$ $\mathrm{sp}(\pi(f); \pi(A))$ is real for all $f \in B_{sa}$.

The property assumed on A implies that $\pi(A)$ is a Banach algebra with dense socle (in the natural quotient norm). Then, by (2.5), the operator norm on $\pi(A)$ is a Q-norm. Every operator in $\pi(A)$ is compact, and so has totally disconnected spectrum. Using (2.3) we infer that $\mathrm{sp}(\pi(g); \pi(A))$ is totally disconnected for all $g \in A$. Let $f = f^* \in A$. Choose $\{f_n\} \subset B$ with $f_n^* = f_n$ for all n and $\|f_n - f\|_A \to 0$. By $(*)$, $\pi(f_n)$ has real spectrum for all n. Also $\pi(f_n) \to \pi(f)$, so that by a result of Newburgh [16], Theorem 3, $\mathrm{sp}(\pi(f); \pi(A))$ is real. But if $\lambda \in \mathrm{sp}(f; A)$, $\lambda \neq 0$, then there exists an algebraically irreducible representation (π, X) of A such that $\lambda \in \mathrm{sp}(\pi(f); \pi(A))$. Therefore, $\mathrm{sp}(f; A)$ is real.

COROLLARY 3.1. *Let $G \in [PG]$ with G compactly generated. Assume that G has a compact subgroup K with the property that every algebraically irreducible representation of $L^1(G)$ is K-finite. Then $L^1(G)$ is symmetric.*

Proof. By [19], Corollary 7, $L^1(G)$ contains a dense symmetric $*$-subalgebra. Then the result follows from Theorem 3.1 and the proof of Theorem 4.5.7.1 in [21].

Corollary 3.1 is more general than Gangolli's result [8], Theorem A. Another symmetry result is given in Corollary 5.2.

4. Locally regular algebras. Let A be a commutative Banach algebra with carrier space Φ ([20], p. 110). The algebra A is regular if, for every closed set $\Gamma \subset \Phi$ and a point $\omega \in \Phi \setminus \Gamma$, there exists an element $f \in A$ such that $\hat{f}(\varphi) = 0$ for all $\varphi \in \Gamma$ and $\hat{f}(\omega) \neq 0$ (algebras with this property are called *completely regular* in [20], p. 174). Regular $*$-algebras have many special properties, some of which are considered in the next lemma. Our interest here in regular algebras is due to the fact that if $G \in [\mathrm{PG}]$ and $f = f^* \in L^1(G) \cap L^2(G)$ has compact support, then $\langle f \rangle$ is regular. This fact is established by an argument due to J. Dixmier which we paraphrase in the proof of (4.1). This local regularity property of $L^1(G)$ has far reaching consequences which we explore in this section. We study a Banach $*$-algebra A with the property that $\langle f \rangle$ is regular for all f in a dense subset of A_{sa}. In important ways, this property determines a correspondence between the ideal theory of A and that of $\bar{A}$ (see Theorem 4.2). Also, it affects the representation theory of the algebra A (see Theorem 5.1).

First we prove a useful lemma concerning properties of commutative regular Banach $*$-algebras.

LEMMA 4.1. *Let A be a commutative regular A^*-algebra. Then*

(1) *γ_A is a unique C^*-norm on A;*

(2) *A is symmetric;*

(3) *any γ-closed ideal of A is a kernel (i.e., an intersection of modular maximal ideals of A);*

(4) *if A is a dense $*$-subalgebra of an A^*-algebra B, then B is regular.*

Proof. Let λ be a C^*-norm on A. By a result of Kaplansky (see [20], Corollary (3.7.7)), $\lambda(f) \geq \varrho(f)$ for all $f \in A$. Therefore, λ is a Q-norm on A. If $f \in A$ and $f = f^*$, then $\lambda(f) = \lambda(f^{2^n})^{2^{-n}} \to \varrho(f)$ by (2.1). Therefore, $\lambda(f) = \varrho(f)$. Of course, the same equality holds for the C^*-norm γ. Thus, for any $f \in A$,

$$\gamma(f)^2 = \gamma(f^*f) = \varrho(f^*f) = \lambda(f^*f) = \lambda(f)^2.$$

This proves (1).

(2) follows from [20], Corollary (3.7.7), and (2.2).

Now let I be a γ-closed ideal of A. Then $\bar{I}$ is a closed ideal in $\bar{A}$ with $I = \bar{I} \cap A$. By [6], Théorème 2.9.5, $\bar{I}$ is the intersection of modular maximal ideals of $\bar{A}$. If M is a modular maximal ideal of $\bar{A}$, then since A is commutative, $A \cap M$ is a modular maximal ideal of A ($A \cap M$ is the kernel of a multiplicative linear functional on A in this case). Thus (3) holds.

Assume that A is a dense subalgebra of a Banach algebra B. Let Φ_A and Φ_B denote the carrier spaces of A and B, respectively. For $\varphi \in \Phi_B$, let φ_0 denote the restriction of φ to A. Since A is dense in B, the map $\varphi \to \varphi_0$ is one-to-one on Φ_B. Let $\psi \in \Phi_A$. By (1), $\gamma_A(f) = \gamma_B(f)$ for all $f \in A$. Then, for some constant K,

$$|\psi(f)| \leq \gamma_A(f) = \gamma_B(f) \leq K \|f\|_B \quad \text{for all } f \in A.$$

Therefore, ψ extends to a multiplicative linear functional on B. This proves that $\varphi \to \varphi_0$ maps Φ_B onto Φ_A. It is easy to see that this map is a homeomorphism. If Γ is a closed subset of Φ_B with $\psi \in \Phi_B \setminus \Gamma$, then there exists an $f \in A$ such that $f(\varphi_0) = 0$ for all $\varphi \in \Gamma$ and $\hat{f}(\psi_0) \neq 0$. This proves that B is regular.

Now we introduce a kind of local regularity property for a Banach $*$-algebra A.

Definition 4.1. Let A be a Banach $*$-algebra. A is *locally regular* if there exists a collection $R \subset A_{sa}$ such that R is dense in A_{sa} and, for each $f \in R$, the algebra $\langle f \rangle$ is regular.

Before deriving the special properties of locally regular algebras, we consider our main example: $L^1(G)$, where $G \in [\mathrm{PG}]$. The basic argument we use here is due to Dixmier [4].

For $z \in C$ (C is the set of the complex numbers) we put

$$u(z) = \exp(iz) - 1.$$

Then $u(z)$ has a power series expansion converging for all $z \in C$ and, therefore, for any $g \in L^1(G)$, $u(g) \in L^1(G)$.

(4.1) *Let A be a compact subset of an LC group G for which there exists a positive integer m such that $|A^n| = O(n^m)$ as $n \to \infty$ (here $|B|$ is the Haar measure of a given set B). Assume that $f = f^* \in L^1(G) \cap L^2(G)$, $\|f\|_1 \leqslant 1$, and* $\mathrm{supp}(f) \subset A$. *Then* $\|u(nf)\|_1 = O(n^{m+1})$.

Straightforward modifications of the arguments of Dixmier in [4], Lemme 6, provide a proof of (4.1).

(4.2) *Assume $G \in [PG]$. Suppose $f = f^* \in L^1(G) \cap L^2(G)$ and f has compact support. Then* $\mathrm{sp}(f; L^1(G))$ *is real and $\langle f \rangle$ is regular.*

Proof. Let A be $L^1(G)$ with identity adjoined. We may assume that $\|f\|_1 = 1$. By (4.1) there exist a positive integer m and a constant $J > 0$ such that $\|u(nf)\|_1 \leqslant Jn^m$ for $n \geqslant 1$. Then $\exp(if) \in A$ and

$$\|(\exp(if))^n\| \leqslant (J+1)n^m \quad \text{for } n \geqslant 1.$$

Thus, $\varrho_A(\exp(if)) \leqslant 1$. Suppose $a+ib \in \mathrm{sp}(f; A)$, where a, b are real with $b \neq 0$. We may assume $b < 0$ (since $a - ib \in \mathrm{sp}(f; A)$). Then

$$\exp(i(a+ib)) = \exp(ia)\exp(-b) \in \mathrm{sp}(\exp(if); A)$$

and

$$\exp(-b) > 1.$$

This contradiction proves that $\mathrm{sp}(f; L^1(G))$ is real.

Now consider $\langle f \rangle$. The carrier space of $\langle f \rangle$ is as usual identified with $\mathrm{sp}(f; L^1(G)) \setminus \{0\}$ or in (some cases with $\mathrm{sp}(f; L^1(G))$). As in [4], Lemme 7, all functions on R with sufficiently many derivatives operate on f. Thus from Lemme 7 in [4] it follows that $\langle f \rangle$ is regular.

(4.3) *Assume $G \in [PG]$ and G is compactly generated. Let*

$$A_0 = \{w \in L^1(G)\colon \exists v \in L^1(G) \text{ such that } v * w = w\}.$$

There exists a fixed positive integer m such that, for every $f = f^*$ in $L^1(G) \cap L^2(G)$ with supp(f) *compact, there exists a sequence $\{w_n\} \subset A_0 \cap \langle f \rangle$ such that*

$$\|w_n - f^m\|_1 \to 0 \quad \text{as } n \to \infty.$$

The result (4.3) is rather technical, but is necessary at a crucial point in the proof of Proposition 5.1. A proof based on [4], Lemme 8, is contained in the proof of [10], Theorem 2.1.

Now let $G \in$ [PG] and assume that G is compactly generated. Let $\omega(x)$ be a polynomial weight on G as in [19], p. 900. Let $L^1(G, \omega)$ be the corresponding L^1-algebra with respect to the weight ω ([19], p. 900). The same arguments as those given above apply in this case with $\omega(x)dx$ in place of Haar measure dx to prove that $L^1(G, \omega)$ is locally regular. These algebras are also always symmetric as shown in [19], Corollary 7.

To summarize:

THEOREM 4.1. *For $G \in [PG]$, $L^1(G)$ is locally regular. If $G \in [PG]$, G is compactly generated, and ω is a polynomial weight on G, then $L^1(G, \omega)$ is locally regular.*

At this point we proceed to establish some basic properties of locally regular A^*-algebras.

LEMMA 4.2. *Let A be a locally regular A^*-algebra. Then γ_A is a unique C^*-norm on A.*

Proof. Let R be as in the definition of a locally regular algebra. Assume that λ is a C^*-norm on A. By Lemma 4.1, γ_A is a unique C^*-norm on $\langle f \rangle$ for $f \in R$. Therefore $\gamma_A(f) = \lambda(f)$ for all $f \in R$. If $g \in A_{sa}$, then choose $\{g_n\} \subset R$ such that $\|g_n - g\| \to 0$. Then $\gamma(g_n - g) \to 0$ and $\lambda(g_n - g) \to 0$. Therefore

$$\gamma(g) = \lim_{n\to\infty} \gamma(g_n) = \lim_{n\to\infty} \lambda(g_n) = \lambda(g).$$

Finally, for any $f \in A$,

$$\gamma(f)^2 = \gamma(f^* f) = \lambda(f^* f) = \lambda(f)^2.$$

It is well known that an LC group G is amenable if and only if the two C^*-norms $\gamma(f)$ and

$$\lambda(f) = \sup\{\|f * g\|_2\colon g \in L^2(G), \|g\|_2 = 1\}$$

coincide for all $f \in L^1(G)$. Thus Theorem 4.1 together with Lemma 4.2 provide a proof of the known fact that if $G \in$ [PG], then G is amenable.

LEMMA 4.3. *Let A be a locally regular A^*-algebra. Let I be a γ-closed ideal of A. Then A/I is locally regular and has a unique C^*-norm.*

Proof. Let $\tilde{R}$ be the set of all $f+I \in A/I$ such that $f \in R$ and $f \notin I$. Certainly, $\tilde{R}$ is dense in $(A/I)_{sa}$. Fix $f \in R$ such that $f \notin I$. By Lemma 4.1, $\langle f\rangle \cap I$ is a kernel in $\langle f\rangle$. This means that there is a closed subset Γ of the carrier space of $\langle f\rangle$ such that

$$\langle f\rangle \cap I = \{g \in \langle f\rangle : \hat{g}(\Gamma) = \{0\}\}.$$

Then by [20], Theorem (3.1.17), the carrier space of $\langle f\rangle/\langle f\rangle \cap I$ is homeomorphic to Γ. From this fact it is straightforward to argue that the algebra $\langle f\rangle/\langle f\rangle \cap I$ is regular. Let φ be the map defined by

$$\varphi(g+\langle f\rangle \cap I) = g+I \qquad (g \in \langle f\rangle).$$

Then φ is a continuous $*$-embedding of $\langle f\rangle/\langle f\rangle \cap I$ into a dense subalgebra of $\langle f+I\rangle$. By Lemma 4.1 (4), $\langle f+I\rangle$ is regular. Thus A/I is locally regular. That A/I has a unique C^*-norm follows from Lemma 4.2.

Concerning Lemma 4.3, when $A = L^1(G)$, $G \in [\mathrm{PG}]$, the technical Banach algebra argument in the proof of this result can be avoided. For in this case for functions f as in the statement of (4.1) we have $\|u(nf)\|_1 \leqslant Jn^{m+1}$ $(n \geqslant 1)$ for some constant $J > 0$ and some positive integer m. The same inequality holds for the quotient norm of $u(n(f+I))$ in A/I. Then the argument in (4.2) shows that $\langle f+I\rangle$ is regular.

Now we turn to the main result of this section.

THEOREM 4.2. *Assume that A is a locally regular A^*-algebra. Let I be a closed ideal in $\bar{A}$. Then $\overline{A \cap I} = I$.*

Proof. Let $J = \overline{A \cap I} \subset I$. By [6], Proposition 1.8.2, both $\bar{A}/I$ and $\bar{A}/J$ are C^*-algebras. Define maps φ_1 and φ_2 on $A/A \cap I$ into $\bar{A}/I$ and $\bar{A}/J$, respectively, by

$$\varphi_1(g+A \cap I) = g+I, \qquad \varphi_2(g+A \cap I) = g+J \qquad (g \in A).$$

Note that both φ_1 and φ_2 are $*$-isomorphisms on $A/A \cap I$. Therefore, the functions

$$\lambda_1(g+A \cap I) = \|\varphi_1(g)\|_{\bar{A}/I}, \qquad \lambda_2(g+A \cap I) = \|\varphi_2(g)\|_{\bar{A}/J} \qquad (g \in A)$$

are C^*-norms on $A/A \cap I$. By Lemma 4.3,

$$(*) \qquad \lambda_1(g+A \cap I) = \lambda_2(g+A \cap I) \qquad (g \in A).$$

Let γ be the norm on $\bar{A}$ ($\gamma(f) = \gamma_A(f)$ for $f \in A$). Assume $f \in I$. Choose $\{f_n\} \subset A$ such that $\gamma(f_n - f) \to 0$. Since $f \in I$,

$$\lambda_1(f_n + A \cap I) = \|f_n + I\|_{\bar{A}/I} = \|(f_n - f) + I\|_{\bar{A}/I} \leqslant \gamma(f_n - f) \to 0.$$

Therefore, by (*),

$$\|f_n+J\|_{\bar{A}/J} = \lambda_2(f_n+A\cap I) \to 0.$$

It follows that there exists $\{h_n\} \subset J$ such that $\gamma(f_n-h_n) \to 0$. Therefore

$$\gamma(f-h_n) \leqslant \gamma(f-f_n)+\gamma(f_n-h_n) \to 0.$$

Thus $f \in J$. This proves $I = J$.

COROLLARY 4.1. *Let A be a locally regular A^*-algebra. If $P \in PRIM^*(A)$, then $\bar{P} \in PRIM(\bar{A})$. If, in addition, A is assumed to be symmetric, then for any $P \in PRIM(A)$ we have $\bar{P} \in PRIM(\bar{A})$.*

Proof. If $P \in \mathrm{PRIM}^*(A)$, then there exists an irreducible *-representation (φ, H) of A such that $P = \ker(\varphi)$. Let $\bar{\varphi}$ be the unique extension of φ to $\bar{A}$. Let $K = \ker(\bar{\varphi})$. Then $P = K\cap A$. By Theorem 4.2, $K = \overline{K\cap A} = \bar{P}$.

The kernel of a continuous irreducible representation of a Banach algebra is a closed ideal with no ideal divisors in the algebra. For this reason, information concerning ideals with this property can prove useful when dealing with the representation theory of the algebra. The next two results concern ideals with no ideal divisors. These results are essentially corollaries to Theorem 4.2.

LEMMA 4.4. *Let A be a locally regular A^*-algebra. Let I be a γ-closed ideal of A such that I has no ideal divisors in A. Then $\bar{I}$ has no ideal divisors in $\bar{A}$.*

Proof. Assume that J and K are ideals in $\bar{A}$ with $JK \subset \bar{I}$. Then $\bar{J}\bar{K} \subset \bar{I}$. Therefore

$$(\bar{J}\cap A)(\bar{K}\cap A) \subset \bar{I}\cap A = I.$$

By hypothesis we have $\bar{J}\cap A \subset I$ or $\bar{K}\cap A \subset I$. Then, by Theorem 4.2,

$$\bar{J} = (\bar{J}\cap A)^- \subset \bar{I} \quad \text{or} \quad \bar{K} = (\bar{K}\cap A)^- \subset \bar{I}.$$

If $\bar{A}$ is either GCR or separable, then every closed ideal of $\bar{A}$ with no ideal divisors is primitive in $\bar{A}$. This follows in the GCR case from [13], Lemma 7.4, and in the separable case from [5], Corollaire 1.

PROPOSITION 4.1. *Let A be a locally regular A^*-algebra such that $\bar{A}$ is either GCR or separable. Assume that P is a γ-closed ideal of A. Then P has no ideal divisors in A if and only if $P \in PRIM^*(A)$.*

Proof. Assume that P is a γ-closed ideal of A with no ideal divisors in A. Then, by Lemma 4.4, $\bar{P}$ has the same property in $\bar{A}$. Consequently, $\bar{P} \in \mathrm{PRIM}(\bar{A})$ (see the remarks preceding the proposition). It follows that $P \in \mathrm{PRIM}^*(A)$.

Let B be a C^*-algebra with structure space Ω (Ω is $\mathrm{PRIM}(B)$ with the hull-kernel topology). In this case B can be represented as an algebra of vector-valued functions on Ω (see [13]). For $f \in B$ and $Q \in \Omega$ we use the notation $\hat{f}(Q) = f + Q \in B/Q$. The norm

$$\|f\|_\infty = \sup\{\|\hat{f}(Q)\|: Q \in \Omega\}$$

is a C^*-norm on B, and therefore $\|f\|_\infty = \|f\|_B$ for all $f \in B$.

In the next result we use the notation above with $\bar{A}$ in place of B.

THEOREM 4.3. *Assume that A is a locally regular A^*-algebra, and let Ω denote the structure space of $\bar{A}$.*

(1) *Let Γ be a closed set in Ω and assume $P_0 \in \Omega \setminus \Gamma$. Then there exists an $f \in A$ such that $\hat{f}(Q) = 0$ for all $Q \in \Gamma$ and $\hat{f}(P_0) \neq 0$.*

(2) *Assume that A is symmetric and Ω is Hausdorff. Let K be a proper closed ideal of A which has no ideal divisors in A. Then K is contained in at most one primitive ideal of A.*

Proof. Let Γ and P_0 be as in (1). Since Γ is hull-kernel closed, there exists a $g \in \bar{A}$ such that $\hat{g}(Q) = 0$ for all $Q \in \Gamma$ and $\hat{g}(P_0) \neq 0$. Then g is contained in the closed ideal $I_\Gamma = \bigcap\{P : P \in \Gamma\}$. By Theorem 4.2, $(A \cap I_\Gamma)^- = I_\Gamma$. Therefore, there exists an $f \in A \cap I_\Gamma$ such that $\hat{f}(P_0) \neq 0$.

Now assume that K is as in (2). Suppose that $K \subset P_1 \cap P_2$, where $P_1, P_2 \in \mathrm{PRIM}(A)$, $P_1 \neq P_2$. By Corollary 4.1, $\bar{P}_1$ and $\bar{P}_2$ are in $\mathrm{PRIM}(\bar{A})$. Let U_1 and U_2 be disjoint open neighborhoods in Ω of $\bar{P}_1$ and $\bar{P}_2$, respectively. For $j = 1, 2$ let

$$I_j = \{f \in A : \hat{f}(Q) = 0 \text{ for } Q \in \Omega \setminus U_j\}.$$

By part (1), $I_j \not\subset P_j$, $j = 1, 2$. But $I_1 I_2 = \{0\} \subset K$, and therefore either I_1 or I_2 is contained in $K \subset P_1 \cap P_2$. This contradiction proves (2).

5. Applications to representation theory. In this section we apply the results of Section 4 to the representation theory of $L^1(G)$, where $G \in [\mathrm{PG}]$. Our main results concern the cases where either G is GCR or G is a Moore group (definition: every irreducible $*$-representation of $L^1(G)$ is finite dimensional). In the latter case we prove that for a Moore group G every continuous irreducible representation of $L^1(G)$ on a normed linear space is finite dimensional (Theorem 5.2). When G is abelian, this is proved by Godement in [9] with the additional hypothesis that every primary ideal in $L^1(G)$ is maximal. Kaplansky [12] verified this hypothesis for a general LC abelian group and extended the result to the case where G is the direct product of a compact group and an abelian group.

We begin with several results concerning the case where $\bar{A}$ is GCR. Let X be a Banach space. Denote by $K(X)$ the algebra of all compact linear operators on X.

LEMMA 5.1. *Let $(J, \|\cdot\|)$ be a Banach algebra of linear operators on X with $J \subset K(X)$. Denote the usual operator norm by $\|\cdot\|_0$. Assume that*

(i) *J is not a radical algebra,*

(ii) $\varrho_J(T) \leqslant \|T\|_0$ $(T \in J)$.

Then there exists a nonzero idempotent in $F(X) \cap J$.

Proof. By hypothesis (ii) the operator norm $\|\cdot\|_0$ is a Q-norm on J. Therefore (2.3) applies, so that for all $T \in J$

$$\operatorname{bndry}(\operatorname{sp}(T; J)) \subset \operatorname{bndry}\big(\operatorname{sp}(T; B(X))\big).$$

Since $J \subset K(X)$, this implies $\operatorname{sp}(T; J) = \operatorname{sp}(T; B(X))$ $(T \in J)$. By (i) there exists an $S \in J$ with $\operatorname{sp}(S; J) \neq \{0\}$. Choose $\lambda \neq 0$, $\lambda \in \operatorname{sp}(S; J)$. Then λ is an isolated point of $\operatorname{sp}(S; J)$ and

$$E = (2\pi i)^{-1} \int_\Gamma (\mu I - S)^{-1} d\mu$$

is a projection in J, where Γ is some circle which contains in its interior λ, but no other point of $\operatorname{sp}(S; J)$. Then $E \neq 0$ and $E \in F(X) \cap J$.

Concerning representations of an algebra, we use the notions FDS ([21], p. 231) and Naĭmark-related ([21], p. 232) as in Warner's book. We note here that an irreducible representation (π, X) is FDS if and only if $\pi(A) \cap F(X) \neq \{0\}$.

THEOREM 5.1. *Let A be a symmetric locally regular A^*-algebra with $\bar{A}$ GCR. Assume that (π, X) is a continuous irreducible Banach representation of A with* $\ker(\pi)$ *γ-closed. Then (π, X) is FDS and Naĭmark-related to an irreducible $*$-representation of A.*

Proof. To prove the theorem it suffices, by [1], Corollary 11, to show that $\pi(A)$ contains a nonzero operator with finite-dimensional range. It is easy to see that if A contains an element e such that $e + \ker(\pi)$ is a nonzero idempotent in $A/\ker(\pi)$ and $\pi(eAe)$ is finite dimensional, then $\pi(e) \in F(X)$.

We proceed to use Lemma 5.1 to establish the existence of such an element e. Let $P = \ker(\pi)$. By Lemma 4.4 and the remarks immediately following Lemma 4.4, $\bar{P} \in \operatorname{PRIM}(\bar{A})$. Since $\bar{A}$ is GCR, there exists an irreducible $*$-representation (φ, H) of $\bar{A}$ with $\ker(\varphi) = \bar{P}$ such that $K(H) \subset \varphi(\bar{A})$. Let I be the closed ideal of $\bar{A}$ defined by $I = \varphi^{-1}\big(K(H)\big)$. Let $J = \varphi(I \cap A)$. By Theorem 4.2, $\overline{I \cap A} = I$, and therefore J is a dense subalgebra of $K(H)$. Since A is symmetric, so is $\varphi(A) \approx A/P$. Hence the operator norm on $\varphi(A)$ (which by Lemma 4.3 is the unique C^*-norm on $\varphi(A)$) is a Q-norm on $\varphi(A)$ by (2.2). Now J is an ideal in $\varphi(A)$, and hence the operator norm is a Q-norm on J. Also, since $\varphi(A)$ is semisimple by [20], Theorem (4.1.19), so is J. We have verified that (i) and (ii) of Lemma 5.1 hold for J. Let E be a nonzero idempotent in $F(H) \cap J$. Choose $e \in A$ such that $\varphi(e) = E$. Then, as argued previously, $\pi(e)$ is a nonzero operator with finite-dimensional range on X. This completes the proof of the theorem.

Corollary 5.1. *Assume that $G \in [PG] \cap [GCR]$, and either $G \in [HER]$ or G is compactly generated. If (π, X) is a continuous irreducible Banach representation of $L^1(G)$ with* $\ker(\pi)$ *γ-closed, then (π, X) is FDS and Naĭmark-related to an irreducible *-representation of $L^1(G)$.*

Proof. In the case where $G \in [\mathrm{HER}]$ the corollary follows directly from the theorem. Assume that G is compactly generated. Then, by [19], Corollary 7, $L^1(G)$ has a dense symmetric *-subalgebra $A = L^1(G, \omega)$, where ω is a polynomial weight. Since $L^1(G, \omega)$ is locally regular (Theorem 4.1), the restriction of the Gelfand-Naĭmark norm γ of $L^1(G)$ to A is equal to γ_A by Lemma 4.2. Therefore, $\tilde{A}$ and $C^*(G)$ can be identified, so that $\tilde{A}$ is GCR. Let (π, X) be as in the statement of the corollary. Let π_0 be the restriction of π to A. By Theorem 5.1, (π_0, X) is FDS and is Naĭmark-related to an irreducible *-representation (φ_0, H) of A. Again, since $\gamma(f) = \gamma_A(f)$ for $f \in A$, φ_0 has a unique extension to a *-representation (φ, H) of $L^1(G)$. Let V be a closed operator, $V: X \to H$ with domain $D(V)$ π_0-invariant and such that

$$V\pi_0(g)x = \varphi_0(g)Vx \qquad (g \in A,\ x \in D(V)).$$

Assume that $f \in L^1(G)$ and $x \in D(V)$. Choose $\{f_n\} \subset A$ such that $\|f_n - f\|_1 \to 0$. Then

$$V\pi_0(f_n)x = \varphi_0(f_n)Vx \to \varphi(f)Vx \qquad \text{and} \qquad \pi_0(f_n)x \to \pi(f)x.$$

Therefore $\pi(f)x \in D(V)$ and $V\pi(f)x = \varphi(f)Vx$. Thus, (π, X) is FDS and Naĭmark-related to the irreducible *-representation (φ, H).

Corollary 5.2. *Assume that $G \in [PG] \cap [GCR]$ and G is compactly generated. Then the following are equivalent:*

(1) *$L^1(G)$ is symmetric;*

(2) *$PRIM(L^1(G)) = PRIM^*(L^1(G))$;*

(3) *if $P \in PRIM(L^1(G))$, then P is γ-closed.*

Proof. We verify that (3) $\Rightarrow$ (1). Let (π, X) be an algebraically irreducible Banach representation of $L^1(G)$. Then $P = \ker(\pi) \in \mathrm{PRIM}(L^1(G))$ and is, by hypothesis, γ-closed. By Corollary 5.1, (π, X) is Naĭmark-related to a *-representation of $L^1(G)$. Then the result follows from [1], Theorem 1.

For a closed ideal J in a Banach algebra A, let

$$h(J) = \{Q \in \mathrm{PRIM}(A) : J \subset Q\} \qquad \text{and} \qquad kh(J) = \bigcap \{Q : Q \in h(J)\}.$$

The next result deals with a certain very special case in which a closed ideal J in $L^1(G)$ has the property that $J = kh(J)$.

Proposition 5.1. *Assume that $G \in [PG]$ and G is compactly generated. Let J be a closed ideal of $L^1(G)$ such that*

(i) *if $Q \in h(J)$, then $L^1(G)/Q$ has finite dimension;*

(ii) *$h(J)$ is finite.*

Then $J = kh(J)$.

Proof. Let $P = kh(J)$ and assume that $J \neq P$. Let C_K be the subspace of $L^1(G)$ consisting of continuous functions with compact support. There is a finite-dimensional subspace D of C_K such that $L^1(G) = D \oplus P$. Let E be the continuous projection of $L^1(G)$ onto P determined by this decomposition. Note that since $D \subset C_K$, we have $E(C_K) \subset C_K$. It follows that $P \cap C_K$ is dense in P. Let

$$P_0 = \{w \in P: \exists v \in P \text{ such that } v * w = w\}.$$

We make the following

CLAIM. *There exists a $w \in P_0$ such that $w \notin J$.*

We establish the Claim by contradiction. Suppose that $P_0 \subset J$. Let m be the fixed positive integer as in (4.3). Then by (4.3) for every $f = f^* \in P \cap C_K$ there exists $\{w_n\} \subset P_0$ such that $\|w_n - f^m\|_1 \to 0$ as $n \to \infty$. Therefore, for every such f, we have $f^m \in J$. But since $C_K \cap P$ is dense in P, it follows that if $g = g^* \in P$, then $g^m \in J$. Now, by [15], Theorem 2, P has a bounded approximate identity $\{u_\lambda: \lambda \in \Lambda\}$, where Λ is a directed set. Then

$$w_\lambda = \left(\tfrac{1}{2}(u_\lambda + u_\lambda^*)\right)^m \quad (\lambda \in \Lambda)$$

is a bounded approximate identity for P, and $w_\lambda \in J$ for all $\lambda \in \Lambda$. Thus $P = J$, contradicting our standing assumption. This proves the Claim.

By the Claim there exists a $w_0 \in P_0$, $w_0 \notin J$. Assume that $v_0 \in P$ and $v_0 * w_0 = w_0$. Let $L_0 = \{f \in P: f * w_0 \in J\}$. Note that $P * (1 - v_0) \subset L_0$ and $J \subset L_0$. Also, since $v_0 * w_0 = w_0 \notin J$, L_0 is proper. Thus there exists a modular maximal left ideal of P that contains L_0, and hence J. Therefore, there exists a primitive ideal Q_0 of P with $J \subset Q_0$. It follows from [7], Proposition 3, that there exists a primitive ideal Q of $L^1(G)$ such that

$$J \subset Q_0 = P \cap Q \subset Q.$$

This is a contradiction.

Assume that G is a compactly generated Moore group and $\mathrm{PRIM}(G)$ is Hausdorff. Let (π, X) be a continuous irreducible representation of $L^1(G)$ on a normed linear space X. Let $J = \ker(\pi)$. The closed ideal J has no ideal divisors in $L^1(G)$. By Theorem 4.3 (2) and [10], p. 295, J is contained in one and only one ideal $P \in \mathrm{PRIM}(L^1(G))$. Therefore, by Proposition 5.1, we have $J = P$. Then, since $L^1(G)/J$ is finite dimensional, so is X. In the original version of this paper we used the remark above as a basis for proving the general result: *if G is a (discrete or compactly generated) Moore group, then every continuous irreducible representation of $L^1(G)$ on a normed linear space is finite dimensional.* While revising this paper we received a preprint *A note on Banach space representations of Moore-groups* by Richard D. Mosak, that essentially contains the same result. Since

Mosak's arguments are shorter, we will simply indicate here how the general result follows by using his ideas.

Let G be a Moore group and assume that (π, X) is a continuous irreducible representation of $L^1(G)$ on the normed linear space X. Set $K = \ker(\pi)$ and let A denote the center of $L^1(G)$. As noted by Mosak, A is a nonzero semisimple regular Tauberian commutative Banach algebra. Assume that $K \cap A$ is contained in two distinct maximal ideals M_1 and M_2 of A. Since A is regular and semisimple, we can choose $g_1, g_2 \in A$ such that $g_1 g_2 = 0$ and $\hat{g}_1(M_1) \neq 0, \hat{g}_2(M_2) \neq 0$. For $k = 1, 2$, let J_k be the closure in $L^1(G)$ of $L^1(G) * g_k$. Then $J_1 \cap J_2 \subset J_1 J_2 = \{0\} \subset K$. Now K has no ideal divisors of zero in $L^1(G)$, and thus $J_1 \subset K$ or $J_2 \subset K$. Then $g_1 \in J_1 \cap A$ or $g_2 \in J_2 \cap A$ is in $K \cap A \subset M_1 \cap M_2$. But this contradicts the choice of g_1 and g_2. Thus $K \cap A$ is contained in at most one maximal ideal of A. From this point the argument proceeds as in the proof of Theorem 1 in Mosak's preprint. This proves the following

THEOREM 5.2. *Assume that G is a Moore group. Then every continuous irreducible representation of $L^1(G)$ on a normed linear space is finite dimensional.*

REFERENCES

[1] B. Barnes, *When is a representation of a Banach *-algebra Naimark-related to a *-representation?*, Pacific Journal of Mathematics (to appear).

[2] J. Boidol, H. Leptin, J. Schürmann and D. Vahle, *Räume primitiver Ideale von Gruppenalgebren*, Mathematische Annalen 236 (1978), p. 1-13.

[3] F. Bonsall and J. Duncan, *Complete normed algebras*, Berlin 1973.

[4] J. Dixmier, *Opérateurs de rang fini dans les représentations unitaires*, Publications Mathématiques de l'Institut des Hautes Etudes Scientifiques 6 (1960), p. 13-25.

[5] — *Sur les C^*-algèbres*, Bulletin de la Société Mathématique de France 88 (1960), p. 95-112.

[6] — *Les C^*-algèbres et leurs représentations*, Paris 1964.

[7] J. Fountain, R. Ramsey and J. H. Williamson, *Functions of measures on compact groups*, Proceedings of the Royal Irish Academy, Section A, 76 (1976), p. 235-251.

[8] R. Gangolli, *On the symmetry of L_1 algebras of locally compact motion groups and the Wiener tauberian theorem*, Journal of Functional Analysis 25 (1977), p. 244-252.

[9] R. Godement, *Théorèmes taubériens et théorie spectrale*, Annales Scientifiques de l'Ecole Normale Supérieure 64 (1948), p. 119-138.

[10] A. Hulanicki, J. W. Jenkins, H. Leptin and T. Pytlik, *Remarks on Wiener's tauberian theorems for groups with polynomial growth*, Colloquium Mathematicum 35 (1976), p. 293-304.

[11] N. Jacobson, *Structure of rings*, American Mathematical Society Colloquium Publications 37 (1956).

[12] I. Kaplansky, *Primary ideals in group algebras*, Proceedings of the National Academy of Sciences of the U.S.A. 35 (1949), p. 133-136.

[13] — *The structure of certain operator algebras*, Transactions of the American Mathematical Society 70 (1951), p. 219-235.

[14] H. Leptin, *Lokal kompakte Gruppen mit symmetrischen Algebren*, Symposia Mathematica 22 (1977), p. 267-281.

[15] T. Liu, A. van Rooij and T. Wang, *Projections and approximate identities for ideals in group algebras*, Transactions of the American Mathematical Society 175 (1973), p. 469-482.

[16] J. Newburgh, *The variation of spectra*, Duke Mathematical Journal 18 (1951), p. 165-176.

[17] T. Palmer, *Hermitian Banach *-algebras*, Bulletin of the American Mathematical Society 78 (1972), p. 522-524.

[18] — *Classes of nonabelian, noncompact, locally compact groups*, Rocky Mountain Journal of Mathematics 8 (1978), p. 683-741.

[19] T. Pytlik, *On the spectral radius of elements in group algebras*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 21 (1973), p. 899-902.

[20] C. Rickart, *Banach algebras*, Princeton, N. J., 1960.

[21] G. Warner, *Harmonic analysis on semi-simple Lie groups, I*, Berlin 1972.

[22] B. Yood, *Homomorphisms on normed algebras*, Pacific Journal of Mathematics 8 (1958), p. 373-381.

[23] — *Ideals in topological rings*, Canadian Journal of Mathematics 16 (1964), p. 28-45.

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