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ANGLES IN METRIC AND NORMED LINEAR SPACES

BY

C. MARTIN AND J. E. VALENTINE (LOGAN, UTAH)

1. Introduction. Menger [6] suggested a system of axioms for "angle spaces". Wilson [9] has shown that if a complete, convex, metric space has each quadruple of its points congruent to a quadruple of points in Euclidean space, then it is possible to develop a theory of angles analogous to that of Euclidean space. However, he also proved [8] that if such a metric space is also externally convex, then it is an inner-product space. Wilson [10] defined angles in arbitrary metric spaces as follows. If a, b, c are points of a metric space, then it follows from the triangle inequality that

$$-1 \leqslant \frac{ab^2+ac^2-bc^2}{2ab\cdot ac} \leqslant 1,$$

where juxtaposition denotes the distance between two points. Thus bac is called an *angle with vertex* a, and its value is given by

$$bac = \arccos\left(\frac{ab^2+ac^2-bc^2}{2ab\cdot ac}\right).$$

It is easily seen that in some metric spaces, even when d is a point between a and b, and e is a point between a and c, there may be no significant relationship between bac and dae. Wilson overcame this objection by defining an angle (ϱ, σ) of two metric rays (congruent images of half-lines) ϱ, σ with common initial point a by $(\varrho, \sigma) = \lim bac$ as b and c tend to a on the rays ϱ and σ, respectively, provided this limit exists. Wilson observed that angles defined in this manner lack many important properties usually associated with angles and he suggested the need for further investigations of the types of spaces admitting these properties and of conditions for the existence of angles.

Valentine and Wayment [7] made a step in this direction. They showed that a normed linear space over the reals is an inner-product space if and only if each pair of rays with a common initial point define an angle. They observed that Blumenthal's example of a convexly metrized tripod shows the result cannot be extended to metric spaces. It should be noted that if ϱ, σ, τ are the three rays of the convexly metrized tripod, then

$$(\varrho, \sigma) = (\varrho, \tau) = (\sigma, \tau) = \pi,$$

and the space does not admit supplementary angles. Thus one is led to conjecture that if M is a complete, convex, externally convex, metric space in which each two intersecting rays determine an angle as defined above and if M admits supplementary angles, then M is an inner-product space. (**P 961**)

James [4] characterized inner-product spaces among the class of real normed linear spaces as those in which Pythagorean orthogonality is homogeneous, that is,

$$\|x\|^2+\|y\|^2 = \|x-y\|^2 \quad \text{implies} \quad \|ax\|^2+\|\beta y\|^2 = \|ax-\beta y\|^2$$

for all real scalars a and β.

Homogeneity of Pythagorean orthogonality is, of course, equivalent to homogeneity of the Euclidean law of cosines for right angles. Looked at in this way, it is possible to remove the right-angle restriction and also to extend the notion to metric spaces. We do that in this paper; thus we show a complete, convex, externally convex, metric space is an inner-product space if and only if the Euclidean law of cosines is homogeneous for some angle not 0 nor π. We also show homogeneity of the cosine law is equivalent to Wilson's definition of angles in the case of a real normed linear space. It would indeed be interesting to know if the same is true for complete, convex, externally convex, metric spaces. The question is the following:

If M is a complete, convex, externally convex, metric space which admits angles and their supplements in the sense of Wilson, is the cosine law homogeneous? (**P 962**)

The answer to this question would also provide an answer to the above-given conjecture. It appears this question is intimately related to whether the local Young property implies the global Young property in the setting of a complete, convex, externally convex, metric space with the two triple property (see [1]). The importance of the Young property lies in its usefulness in showing such a metric space is a normed linear space.

2. Cosine law and k-homogeneity in normed linear spaces. In this section we let A denote a real normed linear space. We define k-homogeneity of the cosine law and show an A having it is an inner-product space.

Definition 2.1. Let k $(-1 \leqslant k \leqslant 1)$ be an arbitrary but fixed real number. The law of cosines is *k-signum homogeneous* provided if x and y

are elements of A with

$$\frac{\|x\|^2+\|y\|^2-\|x-y\|^2}{2\,\|x\|\cdot\|y\|}=k,$$

then

$$\|\alpha x\|^2+\|\beta y\|^2-\|\alpha x-\beta y\|^2=2\,\|\alpha x\|\cdot\|\beta y\|\,k\operatorname{sgn}(\alpha\beta)$$

for each pair of real numbers α, β.

We first prove the following lemma, which shows this definition is non-vacuous:

LEMMA 2.1. *If A has dimension greater than one, x is any non-zero vector in A, and k is any real number ($-1<k<1$), then A contains a vector w such that*

$$\frac{\|x\|^2+\|w\|^2-\|x-w\|^2}{2\,\|x\|\cdot\|w\|}=k.$$

Proof. Let y be any vector such that x, y are linearly independent. If

$$f(\theta)=\frac{\|x\|^2+\|x\cos\theta+y\sin\theta\|^2-\|x(1-\cos\theta)-y\sin\theta\|^2}{2\,\|x\|\cdot\|x\cos\theta+y\sin\theta\|},$$

then f is a continuous function of θ. Moreover, $f(0)=1$ and $f(\pi)=-1$, so there is some θ_0 between 0 and π such that $f(\theta_0)=k$. Let $w=x\cos\theta_0+$ $+y\sin\theta_0$.

It is now possible to show that if the law of cosines is k-signum homogeneous in A for any fixed k ($-1<k<1$), then A is an inner-product space.

THEOREM 2.1. *Let k ($-1<k<1$) be given. If u and v are elements of A such that*

$$\frac{\|u\|^2+\|v\|^2-\|u-v\|^2}{2\,\|u\|\cdot\|v\|}=k$$

implies

$$\|\alpha u\|^2+\|\beta v\|^2-\|\alpha u-\beta v\|^2=2\,\|\alpha u\|\cdot\|\beta v\|\,k\operatorname{sgn}(\alpha\beta)$$

for all real α and β, then A is an inner-product space.

Proof. We show that each pair of vectors x, y of A satisfies the classical Jordan and von Neumann condition, namely,

$$2(\|x\|^2+\|y\|^2)=\|x+y\|^2+\|x-y\|^2. \tag{1}$$

If y is a scalar multiple of x, then, trivially, x, y satisfy (1). Suppose x, y are linearly independent. Using Lemma 2.1, we obtain an element w such that

$$\frac{\|y\|^2+\|w\|^2-\|y-w\|^2}{2\,\|y\|\cdot\|w\|}=k,$$

and w, y are linearly independent. Since the cosine law is k-signum homogeneous in A, we have

$$\|\alpha y\|^2+\|\beta w\|^2-\|\alpha y-\beta w\|^2 = 2\|\alpha y\|\cdot\|\beta w\| k \operatorname{sgn}(\alpha\beta)$$

for each pair of scalars α, β. Since x is in the subspace spanned by y, w, there exist non-zero scalars a, τ such that $x = aw+\tau y$. Thus it follows that

$$\|x\|^2 = \|aw\|^2+\|\tau y\|^2-2\|aw\|\cdot\|\tau y\| k \operatorname{sgn}(a\tau), \tag{2}$$

$$\|x-y\|^2 = \|aw\|^2+\|(\tau-1)y\|^2-2\|aw\|\cdot\|(\tau-1)y\| k \operatorname{sgn}[a(\tau-1)], \tag{3}$$

$$\|x+y\|^2 = \|aw\|^2+\|(\tau+1)y\|^2-2\|aw\|\cdot\|(\tau+1)y\| k \operatorname{sgn}[a(\tau+1)]. \tag{4}$$

Solving (2) for $\|aw\|^2$, substituting the result in (3) and (4), adding the equations so obtained, and simplifying the result, we have (1).

If k is 1 or -1, then the theorem clearly fails, unless the dimension of A is one. Nevertheless, 1-homogeneity of the cosine law in A implies A is rotund.

THEOREM 2.2. *The cosine law is* 1*-homogeneous in* A *if and only if* A *is rotund.*

Proof. If A is rotund, and

$$\frac{\|x\|^2+\|y\|^2-\|x-y\|^2}{2\|x\|\cdot\|y\|} = 1,$$

then $\|x\|-\|y\| = \pm\|x-y\|$. Without loss of generality suppose $\|x\|-\|y\| = \|x-y\|$. Then $\|x-y+y\| = \|x\| = \|x-y\|+\|y\|$ and $x-y$, y are linearly dependent (see [3], p. 111-112). Now $y = \lambda x$ and $0<\lambda<1$ in this case, so

$$\|\alpha x\|^2+\|\beta y\|^2-\|\alpha x-\beta y\|^2 = 2\alpha\beta\lambda\|x\|^2 = 2\|\alpha x\|\cdot\|\beta\lambda x\| \operatorname{sgn}(\alpha\beta),$$

and the cosine law is 1-homogeneous in A.

Conversely, if A is not rotund, it contains linearly independent vectors x, y such that $\|x\|+\|y\| = \|x+y\|$ and $\|x\| = \|y\|$ (see [3], p. 111-112). Now

$$\frac{\|x+y\|^2+\|y\|^2-\|x\|^2}{2\|x+y\|\cdot\|y\|} = 1,$$

but

$$\|x+y\|^2+\|2y\|^2-\|x+y-2y\|^2 = 2\|x+y\|\cdot\|2y\|$$

implies $0 = \|x-y\|$, contrary to the fact that x, y are linearly independent.

3. Cosine law and positive k-homogeneity in normed linear spaces. Throughout this section A denotes a real normed linear space. In Section 2, k-homogeneity was defined for all real scalars α, β. The question naturally arises as to whether just positive scalars would yield the characterization.

Geometrically, homogeneity of the cosine law assures that the angle between αx and βy for all positive α, β is the supplement of the angle between αx and βy for all positive α and for all negative β. Thus the question is, if the angle between αx and βy is well defined for all positive α, β, does it follow that the angle between αx and βy is well defined for all positive α and for all negative β? (**P 963**) The question remains open, but the following example leads one to suspect that positive k-homogeneity is not sufficient to insure that a real normed linear space be an inner-product.

Consider the real normed linear space of ordered pairs of real numbers with usual addition and scalar multiplication and with $\|(a, b)\|$ defined by

$$\|(a, b)\| = \begin{cases} (a^2+b^2)^{1/2} & \text{if } ab \geqslant 0, \\ (|a|^3+|b|^3)^{1/3} & \text{if } ab \leqslant 0. \end{cases}$$

Let $x = (0, 1)$ and $y = (-1, 0)$. Then, for $\alpha > 0$ and $\beta > 0$,

$$\frac{\|\alpha x\|^2 + \|\beta y\|^2 - \|\alpha x - \beta y\|^2}{2\,\|\alpha x\| \cdot \|\beta y\|} = 0.$$

Thus, for this particular choice of x and y, the cosine law is positive 0-homogeneous. However, for $\alpha = 2$ and $\beta = -2$,

$$\frac{\|\alpha x\|^2 + \|\beta y\|^2 - \|\alpha x - \beta y\|^2}{2\,\|\alpha x\| \cdot \|\beta y\|} = \frac{\alpha^2 + \beta^2 - (\alpha^3 + |\beta|^3)^{2/3}}{2\alpha\beta} \neq 0.$$

So the cosine law is not 0-homogeneous.

4. A local characterization of inner product spaces. In this section A will also denote a real normed linear space. Clearly, homogeneity of the cosine law for all angles in A implies the existence of angles between rays according to Wilson's definition. We now show that if certain angles and their supplements exist according to Wilson's definition, then the cosine law is homogeneous for those angles, and hence the space is an inner-product space.

THEOREM 4.1. *A is an inner-product space if and only if there is some k $(-1 < k < 1)$ such that, for each pair of vectors x, y,*

$$\frac{\|x\|^2 + \|y\|^2 - \|x - y\|^2}{2\,\|x\| \cdot \|y\|} = k$$

implies

$$\lim_{\tau,\sigma \to 0} \left[\frac{\|\tau x\|^2 + \|\sigma y\|^2 - \|\tau x - \sigma y\|^2}{2\,\|\tau x\| \cdot \|\sigma y\|} - k\,\mathrm{sgn}(\tau\sigma) \right] = 0.$$

Proof. Since

$$\lim_{\tau,\sigma\to 0}\left[\frac{\|\tau x\|^2+\|\sigma y\|^2-\|\tau x-\sigma y\|^2}{2\,\|\tau x\|\cdot\|\sigma y\|}-k\operatorname{sgn}(|\tau\sigma|)\right]=0,$$

for fixed non-zero τ and σ, we have

$$\lim_{a\to 0}\left[\frac{\|a\tau x\|^2+\|a\sigma y\|^2-\|a\tau x-a\tau y\|^2}{2\,\|a\tau x\|\cdot\|a\sigma y\|}-k\operatorname{sgn}(a^2\tau\sigma)\right]=0.$$

But, clearly, for fixed τ and σ,

$$\lim_{a\to 0}\frac{\|a\tau x\|^2+\|a\sigma y\|^2-\|a\tau x-a\sigma y\|^2}{2\,\|a\tau x\|\cdot\|a\tau y\|}=\frac{\|\tau x\|^2+\|\sigma y\|^2-\|\tau x-\sigma y\|^2}{2\,\|\tau x\|\cdot\|\sigma y\|},$$

which shows the law of cosines is k-signum homogeneous. Theorem 2.1 yields the result.

5. Cosine law and homogeneity in metric spaces. In this section we let M denote a complete, convex, externally convex, metric space.

Definition 5.1. The law of cosines is *homogeneous* in M provided, for each triple of distinct points p, q, r, if q' and r' lie on the metric rays R and R', respectively (where R and R' have initial point p, R passes through q, and R' passes through r), then

$$\frac{pq^2+pr^2-qr^2}{2pq\cdot pr}=\frac{pq'^2+pr'^2-q'r'^2}{2pq'\cdot pr'}.$$

In order to show the homogeneity of the cosine law implies M is an inner-product space, we introduce the Euclidean weak four-point property.

Definition 5.2. A metric space has the *Euclidean weak four-point property* provided each quadruple of its points containing a linear triple is congruent to a quadruple of points in the Euclidean plane.

The importance of the Euclidean weak four-point property lies in its usefulness as a means of characterizing inner-product spaces. Blumenthal [2] has shown that a complete, convex, externally convex, metric space with the weak Euclidean four-point property is an inner-product space. We will show the homogeneity of the cosine law implies the space has the Euclidean weak four-point property.

THEOREM 5.1. *The space M is an inner-product space if and only if the law of cosines is homogeneous in M.*

Proof. It is well known that the law of cosines is homogeneous in an inner-product space. Suppose then that the law of cosines is homogeneous in M. Let p, q, r, s be any quadruple of points of M containing a linear triple, say q, r, s. Without loss of generality assume r is between q and s. The Euclidean plane E_2 contains points p', q', s' such that $p'q' = pq$, $p's' = ps$, and $q's' = qs$. Let r' be the point in E_2 between q' and s' such that $q'r' = qr$ and $r's' = rs$. We show $pr = p'r'$. Since the triple p', q', s' is congruent to the triple p, q, s and the law of cosines is homogeneous in M and E_2, we have

$$\frac{q'p'^2 + q'r'^2 - p'r'^2}{2q'p'\cdot q'r'} = \frac{q'p'^2 + q's'^2 - p's'^2}{2q'p'\cdot q's'} = \frac{qp^2 + qs^2 - ps^2}{2qp\cdot qs}$$
$$= \frac{qp^2 + qr^2 - pr^2}{2qp\cdot qr}.$$

From these equations it is easily seen that $p'r' = pr$. Thus the quadruple p, q, r, s is congruent to the quadruple p', q', r', s' of E_2. By the afore-mentioned result of Blumenthal [2], it now follows that M is an inner-product space.

It should be noted that in this section we have assumed the cosine law is k-homogeneous for each k between -1 and 1, while in the previous sections we only assumed k-homogeneity for a particular k. In the next section we remove the restriction.

6. Cosine law and k-homogeneity in metric spaces. In the last section we characterized inner-product spaces among a certain class of metric spaces by assuming the cosine law is k-homogeneous for each k $(-1 < k < 1)$. It is possible to remove this strong assumption as in Section 2, but apparently one has to postulate some condition to insure "supplementary" angles are really supplementary. We do this as follows.

Definition 6.1. For collinear points a, b, c we define *signum* of the ordered triple (a, b, c) by

$$\operatorname{sgn}(a, b, c) = \begin{cases} 1 & \text{if } c \text{ is between } a \text{ and } b \text{ or } a \text{ is between } b \text{ and } c, \\ 0 & \text{if } a = b,\ b = c \text{ or } a = c, \\ -1 & \text{if } b \text{ is between } a \text{ and } c. \end{cases}$$

Definition 6.2. Let k $(-1 \leqslant k \leqslant 1)$ be an arbitrary but fixed real number. The law of cosines is *k-signum homogeneous* in M provided if p, q, r are distinct elements of M with

$$\frac{pq^2 + pr^2 - qr^2}{2pq\cdot pr} = k,$$

then, for each q' on any ray with initial point p and passing through q, and for each r' on any line passing through p and r,

$$pq'^2 + pr'^2 - q'r'^2 = 2pq'\cdot pr'\, k \operatorname{sgn}(r', p, r).$$

The following lemma shows Definition 6.2 is non-vacuous:

THEOREM 6.1. *If L is a line of M, q a point of M, q not on L, and if $-1 < k < 1$, then L contains points p and r such that*

$$\frac{pq^2+pr^2-qr^2}{2pq\cdot pr} = k.$$

Proof. Let s and t be any distinct points of L with s fixed. Since

$$\frac{st-qs}{2st}+\frac{qt-qs}{2qt}-\frac{qs^2}{2qt\cdot st} \leqslant \frac{qt^2+st^2-qs^2}{2\,qt\cdot st} \leqslant \frac{st+qs}{2st}+\frac{qt+qs}{2\,qt}-\frac{qs^2}{2qt\cdot st}$$

and the extreme sides of these inequalities have limit 1 as $st \to \infty$, we have

$$\lim_{st\to\infty}\frac{qt^2+st^2-qs^2}{2qt\cdot st} = 1.$$

Let u be a fixed point on L such that $us = 1$. We define a real-valued function on L as follows. If t is any point on L, and u is between s and t, then t' is the point on L such that t is between s and t' and $st = tt'$; if t is between s and u, then t' is the point on L such that t is between s and t' and $tt' = 1$, while if $t = s$, then $t' = u$; if s is between t and u and $st \leqslant 1$, then t' is between t and u and $tt' = 1$; if s is between u and t and $st \geqslant 1$, then t' is between t and u and $tt' = st'$. It follows that t' is a continuous function of t. Now let

$$f(t) = \frac{qt^2+tt'^2-qt'^2}{2qt\cdot tt'}.$$

It follows that f is continuous. Now, if s is between t and u and $st \to \infty$, then $f(t) \to 1$. Moreover, routine limit calculations show that if u is between s and t and $st \to \infty$, then $f(t) \to -1$. Thus f assumes every value between -1 and 1.

THEOREM 6.2. *Let k $(-1 < k < 1)$ be given. If u, v, w are elements of M such that*

$$\frac{uv^2+uw^2-vw^2}{2uv\cdot uw} = k$$

implies

$$uv'^2+uw'^2-v'w'^2 = 2uv'\cdot uw'\cdot k\,\mathrm{sgn}(v, u, w)$$

for all v' on any ray with initial point u and passing through v and for all w' on any line passing through u and w, then M is an inner-product space.

Proof. We show each quadruple of points p, q, r, s of M that contains a linear triple is congruent to a quadruple of points of E_2.

If the quadruple p, q, r, s is linear, then it is clearly congruent to a quadruple of E_2. Suppose a triple, say p, q, r, is non-collinear, but p, r, s is a linear triple with, say, r between p and s. Let L be any line which contains p, r, s. Using Theorem 6.1, we find points t, u on L such that

$$\frac{qt^2+tu^2-qu^2}{2qt\cdot tu} = k.$$

Let q', t', u' denote points in E_2 such that the triple q', t', u' is congruent to the triple q, t, u, and let L' denote the line through t' and u'. Let p', r', s' be points on L' such that the quintuple s', r', p', t', u' is congruent to the quintuple s, r, p, t, u. The k-signum homogeneity of the law of cosines in M and E_2 implies immediately that $q'p' = qp$, $q's' = qs$, and $q'r' = qr$. Thus the quadruple p, q, r, s is congruent to p', q', r', s' and, by the result of Blumenthal [2], M is an inner-product space.

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