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ON PRESCRIBED CONNECTIVITIES IN GRAPHS

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The *connectivity* $\varkappa(G)$ of a graph G is the minimum number of points whose removal from G results in either a disconnected graph or the trivial graph having one point. A graph G is *n-connected* if $\varkappa(G) \geqslant n$. The *line-connectivity* $\lambda(G)$ is the smallest number of lines whose removal from G yields a disconnected graph. A graph G is *n-line connected* if $\lambda(G) \geqslant n$. Let $\mu(G)$ denote the minimum degree of the points of G. It is well-known that for any non-trivial connected graph G,

$$0 < \varkappa(G) \leqslant \lambda(G) \leqslant \mu(G).$$

Let G_1 and G_2 be two graphs, where G_i has p_i points and q_i lines, $i = 1, 2$. We say that G_1 is *smaller than* G_2 if either (i) $p_1 < p_2$, or (ii) $p_1 = p_2$ and $q_1 < q_2$. Let b, c, and d be any integers satisfying $0 < b \leqslant c \leqslant d$. The object of this paper is to construct a smallest graph G having

$$\varkappa(G) = b, \quad \lambda(G) = c, \quad \text{and} \quad \mu(G) = d.$$

Let K_p denote the complete graph with p points and $p(p-1)/2$ lines. Denote by $\bar{G}$ the complement of the graph G. The totally disconnected graph on p points is then $\bar{K}_p$. For all connected graphs G we write $2G$ to be the disconnected graph with two components, each isomorphic to G. The *join* of the two graphs G_1 and G_2 is the union of G_1 and G_2 together with all lines of the type v_1v_2, where v_1 is in G_1 and v_2 is in G_2. The join of G_1 and G_2 is denoted by G_1+G_2.

A graph G is *regular of degree r* if every point of G has degree r. We say that G is *almost regular of degree r* if every point of G except one has degree r and the exceptional point has degree $(r+1)$. Let p and r be any two integers with $0 < r < p$. If pr is even, then there exists a regular graph on p points with degree r; while if pr is odd, then there exists an almost regular graph on p points with degree r. (The existence of these graphs is a consequence of König's result [3] on factorization.) We shall denote these graphs by $R_{p,r}$ and $R^*_{p,r}$, respectively.

We now state some results that are necessary for the proofs of this paper.

(A) (Chartrand [1]). If $\mu(G) \geqslant (p-1)/2$ for a graph with p points, then $\mu(G) = \lambda(G)$.

(B) (Chartrand and Harary [2]). If G is a graph with p points and if $\mu(G) \geqslant [(p-2)+n]/2$, where $1 \leqslant n \leqslant p-1$, then G is n-connected.

(C) (Menger's Theorem [4]). A graph G is n-connected (n-line connected) if and only if any two distinct points of G are joined by at least n disjoint (line-disjoint) paths.

THEOREM 1. *For any integers b and c satisfying $0 < b \leqslant c$, there exists a smallest graph G with $\varkappa(G) = b$ and $\lambda(G) = c$. Furthermore, a smallest such graph is K_{b+1} when $b = c$; is $G_{b,c} = 2K_{c-b+1} + \bar{K}_b$ when $c \geqslant 2(b-1)$ and $c > b$; is $G_{b,c} = 2K_{c-b+1} + R_{b,2b-c-2}$ when $b < c < 2(b-1)$ and bc is even; and is $G_{b,c} = 2K_{c-b+1} + R^*_{b,2b-c-2}$ when $b < c < 2(b-1)$ and bc is odd.*

Proof. Chartrand and Harary [2] proved that a graph G^* with $\varkappa(G^*) = b$, $\lambda(G^*) = c$, and with the minimal number of points is K_{b+1} if $b = c$; and if $b < c$, then G^* is necessarily of the form $2K_{c-b+1} + H_b$, where H_b is a certain graph with b points. In their construction they let $H_b = K_b$.

Now, if $b = c$, then $G = K_{b+1}$, since any smaller graph has $\varkappa \leqslant \lambda \leqslant \mu < b$. Assume $b < c$. Using (A) and (B) similarly as in [2] one obtains that $G = 2K_{c-b+1} + H_b$ if H_b is the smallest graph such that $\mu(2K_{c-b+1} + H_b) = c$. It is easily seen that the graphs in Theorem 1 have this property.

A few examples of the graphs $G_{b,c}$ are shown in Fig. 1.

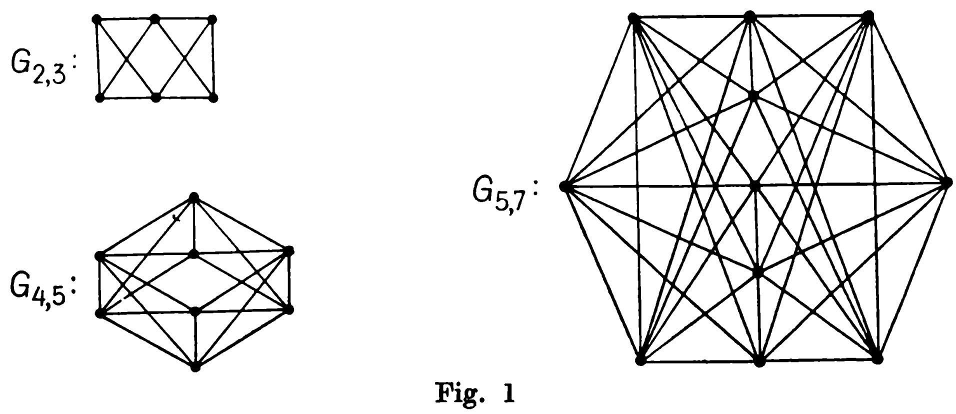

Fig. 1

THEOREM 2. *Let b, c, and d be any integers such that $0 < b \leqslant c \leqslant d$. Then there exists a smallest graph G with*

$$\varkappa(G) = b, \quad \lambda(G) = c, \quad \text{and} \quad \mu(G) = d.$$

Furthermore, a smallest such graph is constructed in the proof.

Proof. Chartrand and Harary [2] proved that a graph G^* with $\varkappa(G^*) = b$, $\lambda(G^*) = c$, $\mu(G^*) = d$, and with a minimum number of points is given in Theorem 1 if $c = d$; and if $c < d$, then G^* must have $2(d+1)$ points. In their construction, they use two copies of K_{d+1} with c lines properly added so that the resulting graph has the desired connectivities and minimum degree. Fig. 2 provides an example which shows that their construction does not yield the smallest such graph in terms of the number of points and lines.

Fig. 2

We now alter the above construction to give the smallest such graph in terms of the number of points and lines. Since the smallest graph with $\varkappa = b$, $\lambda = c$, and $\mu = d$ in terms of points has $2(d+1)$ points, the smallest graph in terms of points and lines must have $2(d+1)$ points, and since the minimum degree is d, there must be at least $d(d+1)$ lines.

Because the construction varies for different values of b, c, and d, we give the construction in five cases.

Case 1. *b and c both even.* Let $H_1 = H_2 = K_{d-k+1}$ and $H_3 = H_4 = K_k$, where $b = 2k$ and $c = 2t$. The degree of each point of H_1 and H_2 is $d-k$ and the degree of each point of H_3 and H_4 is $k-1$. Label the points of H_1 by $u_1, u_2, \ldots, u_{d-k+1}$; the points of H_2 by $v_1, v_2, \ldots, v_{d-k+1}$; the points of H_3 by $w_1, w_2, \ldots, w_k$; and the points of H_4 by $x_1, x_2, \ldots, x_k$. To the graph $H_1 \cup H_2 \cup H_3 \cup H_4$ add the lines $w_1u_1, w_2u_2, \ldots, w_ku_k, w_ku_{k+1}, \ldots$ $\ldots, w_ku_t, x_1v_1, x_2v_2, \ldots, x_kv_k, x_kv_{k+1}, \ldots, x_kv_t$. Next add the lines w_iv_j and x_iu_j, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant d-k+1$, if the lines w_iu_j and x_iv_j are not added. In adding these $2k(d-k+1)$ lines, we have added the exact number of lines in such a manner to make the resulting graph regular of degree d. Hence $\mu = d$. Since the removal of the points of H_3 and H_4 disconnects the graph, $\varkappa \leqslant 2k = b$. Since the removal of the t lines between H_2 and H_4 and the removal of the t lines between H_1 and H_3 disconnects the graph, $\lambda \leqslant 2t = c$. It is now possible to construct $2k = b$ disjoint paths between any two points of the graph and so by (C), $\varkappa \geqslant 2k = b$. Also, it is possible to construct $2t = c$ line-disjoint paths between any two points of the graph and so by (C), $\lambda \geqslant 2t = c$. Hence this graph has $\varkappa = b$, $\lambda = c$, $\mu = d$, and is a smallest such graph possible.

Case 2. *$1 < b$, b odd and c even.* Let $b = 2k+1$ and $c = 2t$. Let $H_1 = K_{d-k}$, $H_2 = K_{d-k+1}$, $H_3 = K_k$, and $H_4 = K_{k+1}$. The degree of each point of H_1 is $d-k-1$, of H_2 is $d-k$, of H_3 is $k-1$, and of H_4 is k. Label the points of H_1 by $u_1, u_2, \ldots, u_{d-k}$, of H_2 by $v_1, v_2, \ldots, v_{d-k+1}$, of H_3 by $w_1, w_2, \ldots, w_k$, and of H_4 by $x_1, x_2, \ldots, x_{k+1}$. To the graph $H_1 \cup H_2 \cup H_3 \cup H_4$ add the lines $w_1u_1, w_2u_2, \ldots, w_ku_k, w_ku_{k+1}, \ldots, w_ku_t$, $x_1v_1, x_2v_2, \ldots, x_{k+1}v_{k+1}$, $x_{k+1}v_{k+2}, \ldots, x_{k+1}v_t$. Next, add all the lines w_iv_j and $x_\varrho u_\sigma$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant d-k+1$, $1 \leqslant \varrho \leqslant k+1$, $1 \leqslant \sigma \leqslant d-k$, if the lines w_iu_j and $x_\varrho v_\sigma$ are not added. In adding these $2k(d-k)+d$ lines, we have added the exact number of lines in such a manner to make the resulting graph regular of degree d. Hence $\mu = d$. That $\varkappa = b$ and $\lambda = c$ is easily verified.

Before considering Case 3, we show that if c is odd, then the smallest graph having $\varkappa = b$, $\lambda = c$, and $\mu = d > c$ is not regular. In order to verify this we assume that G is a regular graph of degree d having at most $2(d+1)$ points, $\varkappa = b$, and $\lambda = c$, where c is odd. Since $\lambda = c$, there exist c lines whose removal disconnects G and there are necessarily two components, say L_1 and L_2, when c such lines are removed. We claim that each of L_1 and L_2 has $d+1$ points, for suppose to the contrary that L_1 has k points, where $k \leqslant d$. Every point of L_1 is then adjacent to at most $k-1$ other points of L_1. Since each of the points of L_1 is adjacent to at least $d+1-k$ points of L_2 and exactly c lines join L_1 with L_2, we have $k(d+ +1-k) \leqslant c$. The minimum value of $k(d+1-k)$ is d, which occurs if k is d or 1. Since $d > c$, this is a contradiction so that L_1 and L_2 each have $d+1$ points. The sum of the degrees of the points of L_1 in G is $d(d+1)$ so that the sum of the degrees of L_1 considered as a graph (i.e., the subgraph induced by the points of L_1) is $d(d+1)-c$. Since c is odd, $d(d+1)-c$ is odd, which contradicts the fact that the sum of the degrees of a graph is always even.

Hence there exists no regular graph with the afore mentioned properties.

Case 3. $1 < b = c$, *c odd*. Let $H_1 = H_2 = K_{d+1}$, and label the points of H_1 by $u_1, u_2, \ldots, u_{d+1}$, and label the points of H_2 by $v_1, v_2, \ldots, v_{d+1}$. To the graph $H_1 \cup H_2$ add the lines $u_1v_1, u_2v_2, \ldots, u_cv_c$ and delete the lines $u_1u_2, u_3u_4, \ldots, u_{c-2}u_{c-1}, v_2v_3, v_4v_5, \ldots, v_{c-1}v_c$. The resulting graph has the properties $\varkappa = b = c = \lambda$, and $\mu = d$. This graph is a smallest graph having the prescribed connectivities since such a graph with c odd cannot be regular and this graph has exactly one more line than the corresponding regular graph.

Case 4. $1 < b \neq c$ *and c odd*. By either Case 1 or Case 2 we construct a smallest regular graph H with degree d, $2(d+1)$ points, and $d(d+1)$ lines having $\varkappa = b$ and $\lambda = c-1$. To the graph H we add the line x_1v_2, where x_1 and v_2 are the vertices defined in Cases 1 and 2. The resulting graph now has $\varkappa = b$, $\lambda = c$, and $\mu = d$. Since c is odd, the smallest graph having these properties must have at least $d(d+1)+1$ lines and the graph just constructed has exactly so many lines.

Case 5. $b = 1$. In this case, it is readily verified that every point but one must have degree at least d while the exceptional point must have degree at least $d+c$. Hence the minimum number of lines in a graph with these prescribed connectivities and minimum degree must be $d(d+1)+\{c/2\}$, where $\{x\}$ denotes the least integer not less than x. We now construct such a graph. Let $H_1 = H_2 = K_{d+1}$. In H_1 label one point by v and in H_2 label c points by $u_1, u_2, \ldots, u_c$. To $H_1 \cup H_2$ add the lines $vu_1, vu_2, \ldots, vu_c$. If c is even, delete the lines $u_1u_2, u_3u_4, \ldots, u_{c-1}u_c$. If c is odd, delete the lines $u_1u_2, u_3u_4, \ldots, u_{c-2}u_{c-1}$. The graph so constructed has the desired number of lines.

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