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AN EXTENSION OF THE STEINHAUS–WEIL THEOREM

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Consider a locally compact topological group with some Haar measure and two subsets A and B of finite positive measure. Then, according to a theorem of Weil ([10], p. 50), the set $A \cdot B = f(A \times B)$ has non-empty interior, where f is the group operation. The special case of the additive group of real numbers is due to Steinhaus [9].

It seems natural to seek corresponding results for general binary mappings f on topological measure spaces. A first such result was obtained by Erdös and Oxtoby [2] who considered continuously differentiable functions on open subsets of $\boldsymbol{R} \times \boldsymbol{R}$ with non-vanishing partial derivatives, thus generalizing Steinhaus' theorem. This inspired Kuczma [5] to impose, in the case of abstract binary mappings f on topological measure spaces, some Radon–Nikodým type differentiability with respect to both variables together with global solvability of the equation $z = f(x, y)$. (Further Steinhaus type theorems can be found in [6], [4] and [7].)

Kuczma's theorem, however, does not contain Weil's theorem in full generality. Moreover, his assumption of global solvability appears rather strong; he asks ([5], p. 106) whether it can be weakened to local solvability. The aim of this paper is to answer this question in the affirmative, providing a Steinhaus type theorem that is strong enough to comprise Weil's theorem. Our main tool is, as in [5], an abstract rule of differentiation of implicit functions: we prove a local version of Kuczma's Proposition 2. A Fubini theorem for Radon measures on arbitrary Hausdorff topological spaces, due to Schwartz [8], allows us to drop several measurability assumptions that were made by Kuczma.

1. Notation and definitions. A triple (X, Σ, λ) consisting of a Hausdorff topological space X, a σ-algebra Σ of subsets of X and a measure λ defined on Σ is called a *topological measure space* if Σ contains every open set, i.e., if Σ contains $\mathscr{B}(X)$, the σ-algebra of *Borel sets* generated by the open sets in X. The measure λ is then called a *topological measure.* A topological measure λ is *regular* if

$$\lambda(E) = \inf\{\lambda(O): O \supset E \text{ open}\} \quad \text{for all } E \in \Sigma$$

and

$$\lambda(E) = \sup\{\lambda(K): K \subset E \text{ compact}\} \quad \text{for all open sets } E$$

(and hence for all $E \in \Sigma$ with $\lambda(E) < \infty$; cf. [3], (11.32)). A regular topological measure λ with $\lambda(K) < \infty$ for all compact sets K is called a *Radon measure*, and the corresponding triple (X, Σ, λ) (or (X, λ) for short) a *Radon measure space*. A *Haar measure* is a left or right invariant Radon measure on a (necessarily locally compact) topological group. For the theory of Radon measures see [8], and for Haar measures see [3].

For a topological measure space (X, Σ, λ) and a topological space Y, a mapping g: $X \to Y$ is called *measurable* if $g^{-1}(O) \in \Sigma$ for all open sets O in Y, in particular *Borel measurable* if $\Sigma = \mathscr{B}(X)$. In the case of $Y = \boldsymbol{R}$ we understand (λ-) *integrability* in the sense of abstract measure theory. (Schwartz' "λ-strict integrability" in [8] is integrability with respect to the completion of λ.)

For topological measure spaces (X, Σ, λ) and (Y, T, μ), a measurable mapping ϱ: $X \to \boldsymbol{R}_0^+$ is called the *Radon–Nikodým derivative* (*RN-derivative*) of a mapping g: $X \to Y$ if

$$g(A) \in T \quad \text{and} \quad \mu(g(A)) = \int_A \varrho \, d\lambda$$

for all Borel sets A in X (but not necessarily for all $A \in \Sigma$).

Consider now topological spaces X, Y and Z and a mapping f: $D \to Z$, where D is an open subset of $X \times Y$. We say that f is *continuously solvable* at a point $(x_0, y_0) \in D$ if there are neighbourhoods U of x_0, V of y_0 and W of $z_0 = f(x_0, y_0)$ with $U \times V \subset D$ and continuous mappings

$$\varphi: U \times W \to Y \quad \text{and} \quad \psi: V \times W \to X$$

such that for all $(x, y, z) \in U \times V \times W$ we have

$$f(x, y) = z \Leftrightarrow \varphi(x, z) = y \Leftrightarrow \psi(y, z) = x.$$

If λ, μ and ν are topological measures on X, Y and Z, respectively, then we call f *RN-differentiable* at $(x_0, y_0) \in D$ if there are open neighbourhoods U of x_0 and V of y_0 with $U \times V \subset D$ and mappings

$$\alpha, \beta: U \times V \to \boldsymbol{R}_0^+$$

such that for each $x \in U$ and $y \in V$ the mappings $f_x|_V$ and $f^y|_U$ have RN-derivatives β_x and α^y, respectively. We call α and β the *partial RN-derivatives* of f.

Above, $\boldsymbol{R}_0^+$ denotes the set of non-negative real numbers. For a mapping f: $X \times Y \to Z$ the mappings f_x: $Y \to Z$ and f^y: $X \to Z$ are defined by

$$f_x(y) = f^y(x) = f(x, y) \quad \text{for } x \in X \text{ and } y \in Y.$$

id always denotes the identity mapping.

We adopt D. H. Fremlin's **P**...**Q** notation to mark off the proof of the statement immediately preceding **P**.

2. A rule of differentiation of implicit functions for RN-differentiable mappings. In this section we prove a local version of Kuczma's Proposition 2. Apart from arguments similar to Kuczma's we need two auxiliary results.

In abstract measure theory, the product of two measure spaces is endowed with a measure defined on the smallest σ-algebra containing all measurable rectangles; the Fubini theorem holds for this measure. Now, in the product of two topological measure spaces the Borel sets constitute a more natural σ-algebra that is in general larger than the product σ-algebra. So, the classical Fubini theorem is not good enough to deal with this situation. Instead, we need:

THEOREM A (Schwartz [8], pp. 63–70). *Let (X, λ) and (Y, μ) be finite and complete Radon measure spaces. Then there is a Radon measure $\lambda\times\mu$ on $X\times Y$ such that*

$$(\lambda\times\mu)(A\times B) = \lambda(A)\cdot\mu(B)$$

for all Borel subsets A of X and B of Y. For any $(\lambda\times\mu)$-integrable function $\gamma\colon X\times Y\to \mathbf{R}$ the function γ^y is λ-integrable for μ-almost all $y\in Y$, the mapping

$$y\mapsto \int \gamma(x, y)\,d\lambda(x)$$

is μ-integrable and

$$\int \gamma\, d(\lambda\times\mu) = \int\Big(\int \gamma(x, y)\,d\lambda(x)\Big)d\mu(y).$$

The following simple result allows us to avoid talking about inverse mappings.

LEMMA B. *Let X, Y and Z be topological spaces and $f\colon X\to Y$, $g\colon Y\to Z$ and $h\colon X\to Z$ mappings with $h = g\circ f$. If h is an open mapping, then*

(a) *f is open if g is continuous and injective;*

(b) *g is open if f is continuous and surjective.*

Proof. If g is injective, then

$$f(E) = g^{-1}(h(E)) \quad \text{for } E\subset X,$$

and if f is surjective, then

$$g(E) = h(f^{-1}(E)) \quad \text{for } E\subset Y.$$

We can now prove

THEOREM 1. *Let (X, λ), (Y, μ) and (Z, ν) be Radon measure spaces, D an open subset of $X\times Y$, $(x_0, y_0)\in D$, and $f\colon D\to Z$ a continuous mapping. Put $z_0 = f(x_0, y_0)$. Suppose that*

(i) *f is continuously solvable at (x_0, y_0) with solutions φ and ψ;*

(ii) *f is continuously RN-differentiable at (x_0, y_0) with non-vanishing partial RN-derivatives α and β;*

(iii) *there is a neighbourhood $\tilde{W}$ of z_0 such that $\nu(W') > 0$ for all open non-empty subsets W' of $\tilde{W}$.*

Then there are open neighbourhoods U of x_0, V of y_0, and W of z_0 with $U \times V \subset D$ such that for all $z \in W$ the mappings

$$x \mapsto \frac{\alpha(x, \varphi(x, z))}{\beta(x, \varphi(x, z))} \quad \text{and} \quad y \mapsto \frac{\beta(\psi(y, z), y)}{\alpha(\psi(y, z), z)}$$

$(x \in U,\ y \in V)$ are the RN-derivatives of $\varphi^z|_U$ and $\psi^z|_V$, respectively.

Proof. By symmetry it suffices to prove the assertion for φ. We divide the proof into several steps.

(1) Since λ, μ and ν are Radon measures and α and β are continuous, we can find neighbourhoods U_0 of x_0, V_0 of y_0, and W_0 of z_0 of finite measure with $U_0 \times V_0 \subset D$ in which the conditions of (i)–(iii) hold and such that there are constants δ, $M > 0$ with

$$\delta \leqslant \alpha(x, y) \leqslant M \quad \text{and} \quad \delta \leqslant \beta(x, y) \leqslant M \quad \text{for } x \in U_0 \text{ and } y \in V_0.$$

In this first step we investigate the properties of the mappings

$$F\colon U_0 \times V_0 \to U_0 \times Z, \quad F(x, y) = (x, f(x, y));$$
$$G\colon U_0 \times V_0 \to Z \times V_0, \quad G(x, y) = (f(x, y), y);$$
$$\Phi\colon U_0 \times W_0 \to W_0 \times Y, \quad \Phi(x, z) = (z, \varphi(x, z));$$

as well as f_x, f^y and φ^z.

As auxiliary mappings we consider

$$H\colon U_0 \times W_0 \to U_0 \times Y, \quad H(x, z) = (x, \varphi(x, z))$$

and

$$J\colon W_0 \times V_0 \to X \times V_0, \quad J(z, y) = (\psi(y, z), y).$$

(1a) Since f, φ and ψ are continuous mappings on $U_0 \times V_0$, $U_0 \times W_0$ and $V_0 \times W_0$, respectively, any of the mappings above is continuous, and there are open neighbourhoods $U_2 \subset U_1 \subset U_0$ of x_0, $V_2 \subset V_1 \subset V_0$ of y_0, and $W_2 \subset W_1 \subset W_0$ of z_0 such that

$$\varphi(U_1 \times W_1) \subset V_0, \quad \psi(V_1 \times W_1) \subset U_0,$$
$$f(U_1 \times V_1) \subset W_1 \quad \text{and} \quad \varphi(U_2 \times W_2) \subset V_1.$$

(1b) It is now easy to deduce from (i) that the following are injective: F and G on $U_1 \times V_1$, Φ and H on $U_1 \times W_1$, J on $W_1 \times V_1$, f^y and φ^z on U_1, f_x and ψ^z on V_1, and φ_x and ψ_y on W_1 for $x \in U_1$, $y \in V_1$ and $z \in W_1$. Moreover, we have

$$U_2 \times W_2 \subset F(U_1 \times V_1).$$

(1c) Similarly, it follows from (i) that

$$H \circ F = J \circ G = \mathrm{id} \quad \text{and} \quad \Phi \circ F = G \quad \text{on } U_1 \times V_1,$$
$$\varphi_x \circ f_x = \mathrm{id} \quad \text{on } V_1 \text{ for } x \in U_1,$$
$$\psi_y \circ f^y = \mathrm{id} \quad \text{on } U_1 \text{ for } y \in V_1$$

and

$$\psi^z \circ \varphi^z = \mathrm{id} \quad \text{on } U_2 \text{ for } z \in W_2,$$

where all the composite mappings are well defined.

(1d) Applying Lemma B we infer from (1a)-(1c) that F, G, Φ, $f_x (x \in U_1)$, $f^y (y \in V_1)$ and $\varphi^z (z \in W_2)$ are homeomorphisms on $U_1 \times V_1$, $U_1 \times V_1$, $U_2 \times W_2$, V_1, U_1 and U_2, respectively.

The properties of (1d) ensure, in particular, that the expressions considered in the remainder of the proof are well defined. Let in the following $\hat{\lambda}$, $\hat{\mu}$ and $\tilde{\nu}$ denote the completions of the measures λ, μ and $\nu|_{\mathscr{B}(Z)}$, the restriction of ν to $\mathscr{B}(Z)$.

(2) In this step we prove that for all Borel subsets A of U_2 we have

$$\mu(\varphi^z(A)) = \int_A \frac{\alpha(x, \varphi(x, z))}{\beta(x, \varphi(x, z))} d\lambda(x)$$

for ν-almost all $z \in W_2$.

(2a) For all Borel subsets Q of $U_1 \times V_1$ we have

$$(\hat{\lambda} \times \tilde{\nu})(F(Q)) = \int_Q \beta d(\hat{\lambda} \times \hat{\mu}) \quad \text{and} \quad (\tilde{\nu} \times \hat{\mu})(G(Q)) = \int_Q \alpha d(\hat{\lambda} \times \hat{\mu}).$$

P Since

$$F(U_1 \times V_1) \subset U_0 \times W_0$$

with

$$(\hat{\lambda} \times \tilde{\nu})(U_0 \times W_0) < \infty, \quad (\hat{\lambda} \times \hat{\mu})(U_0 \times V_0) < \infty$$

and

$$\beta(x, y) \leqslant M \quad \text{for } (x, y) \in U_0 \times V_0,$$

we can apply Theorem A twice to derive for any Borel subsets A of U_1 and B of V_1:

$$\begin{aligned}(\hat{\lambda} \times \tilde{\nu})(F(A \times B)) &= (\hat{\lambda} \times \tilde{\nu})(\{(x, z) \colon x \in A \text{ and } z \in f_x(B)\}) \\ &= \int_A \nu(f_x(B)) d\hat{\lambda}(x) = \int_A \Big(\int_B \beta_x d\mu\Big) d\hat{\lambda}(x) = \int_{A \times B} \beta d(\hat{\lambda} \times \hat{\mu}).\end{aligned}$$

Note here that $f_x(B)$ is a Borel set in Z for all $x \in U_1$ by (1d). Hence the claim holds for all finite unions of open rectangles. Now let O be any open subset of $U_1 \times V_1$. Let $\varepsilon > 0$ and let K be a compact subset of O with

$$(\hat{\lambda} \times \tilde{\nu})(F(O) \backslash F(K)) < \varepsilon \quad \text{and} \quad (\hat{\lambda} \times \hat{\mu})(O \backslash K) < \varepsilon.$$

Then there is a finite union $\tilde{O}$ of open rectangles with $K \subset \tilde{O} \subset O$. Hence

$$\begin{aligned}\int_O \beta d(\hat{\lambda} \times \hat{\mu}) - M \cdot \varepsilon \leqslant \int_{\tilde{O}} \beta d(\hat{\lambda} \times \hat{\mu}) &\leqslant (\hat{\lambda} \times \tilde{\nu})(F(O)) \\ &\leqslant (\hat{\lambda} \times \tilde{\nu})(F(\tilde{O})) + \varepsilon \leqslant \int_O \beta d(\hat{\lambda} \times \hat{\mu}) + \varepsilon.\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, the claim now holds for all open subsets of $U_1 \times V_1$, and hence, by a standard argument, for all Borel subsets. The claim for G follows similarly. **Q**

(2b) For all Borel subsets Q of $U_1 \times V_1$ and all Borel measurable functions γ: $U_1 \times W_1 \to \boldsymbol{R}_0^+$ we have (in $[0, \infty]$)

$$\int_{F(Q)} \gamma d(\hat{\lambda} \times \tilde{\nu}) = \int_Q (\gamma \circ F) \cdot \beta d(\hat{\lambda} \times \hat{\mu}).$$

P If R is a Borel subset of $U_1 \times W_1$ and χ_R its characteristic function, then by (2a) we have

$$\begin{aligned} \int_Q (\chi_R \circ F) \cdot \beta d(\hat{\lambda} \times \hat{\mu}) &= \int_{F^{-1}(R) \cap Q} \beta d(\hat{\lambda} \times \hat{\mu}) = (\hat{\lambda} \times \tilde{\nu})(F(F^{-1}(R) \cap Q)) \\ &= (\hat{\lambda} \times \tilde{\nu})(F(Q) \cap R) = \int_{F(Q)} \chi_R d(\hat{\lambda} \times \tilde{\nu}). \end{aligned}$$

Hence the assertion holds for all simple Borel measurable functions, and therefore for all γ. **Q**

(2c) For all Borel subsets R of $U_2 \times W_2$ we have

$$(\tilde{\nu} \times \hat{\mu})(\Phi(R)) = \int_R \frac{\alpha \circ H}{\beta \circ H} d(\hat{\lambda} \times \tilde{\nu}).$$

P Applying (2b) to the positive continuous function

$$\gamma = \frac{\alpha \circ H}{\beta \circ H},$$

we obtain for all Borel subsets Q of $U_1 \times V_1$, using (1c) and (2a),

$$\begin{aligned} \int_{F(Q)} \frac{\alpha \circ H}{\beta \circ H} d(\hat{\lambda} \times \tilde{\nu}) &= \int_Q \frac{\alpha \circ H \circ F}{\beta \circ H \circ F} \cdot \beta d(\hat{\lambda} \times \hat{\mu}) = \int_Q \frac{\alpha}{\beta} \cdot \beta d(\hat{\lambda} \times \hat{\mu}) \\ &= (\tilde{\nu} \times \hat{\mu})(G(Q)) = (\tilde{\nu} \times \hat{\mu})(\Phi(F(Q))). \end{aligned}$$

Now, since F is a homeomorphism on $U_1 \times V_1$ and

$$U_2 \times W_2 \subset F(U_1 \times V_1),$$

the assertion follows. **Q**

(2d) We can now prove (2): Consider Borel subsets A of U_2 and C of W_2. Theorem A implies

$$\begin{aligned} (\tilde{\nu} \times \hat{\mu})(\Phi(A \times C)) &= (\tilde{\nu} \times \hat{\mu})(\{(z, y)\colon z \in C \text{ and } y \in \varphi^z(A)\}) \\ &= \int_C \mu(\varphi^z(A)) d\tilde{\nu}(z). \end{aligned}$$

Note here that $\varphi^z(A)$ is a Borel set in Y for all $z \in W_2$ by (1d). On the other hand, by (2c) we have

$$(\tilde{\nu} \times \hat{\mu})(\Phi(A \times C)) = \int_C \left(\int_A \frac{\alpha \circ H}{\beta \circ H} d\lambda \right) d\tilde{\nu}.$$

Hence the two right-hand sides coincide for all Borel subsets C of W_2, thus also for all $\tilde{\nu}$-measurable subsets. This implies that the two integrands agree $\tilde{\nu}$-almost everywhere in W_2, whence (2) holds.

(3) In this final step we use the continuity assumptions on α, β, φ and ψ to show that in (2) equality holds everywhere in W_2, which is the assertion of the theorem. For convenience, we put

$$\gamma = \frac{\alpha \circ H}{\beta \circ H} \quad \text{on } U_2 \times W_2.$$

(3a) For all compact subsets K of U_2, all $z \in W_2$ and $\varepsilon > 0$ there is a neighbourhood $W' \subset W_2$ of z with

$$\mu(\varphi^z(K)) \geqslant \mu(\varphi^\zeta(K)) - \varepsilon \quad \text{for } \zeta \in W'.$$

P By regularity of μ, there is an open subset V' of V_0 with

$$\varphi^z(K) \subset V' \quad \text{and} \quad \mu(V' \setminus \varphi^z(K)) < \varepsilon.$$

Since φ is continuous and K is compact, there is a neighbourhood $W' \subset W_2$ of z with

$$\varphi^\zeta(K) \subset V' \quad \text{for } \zeta \in W',$$

whence

$$\mu(\varphi^\zeta(K)) \leqslant \mu(V') \leqslant \mu(\varphi^z(K)) + \varepsilon \quad \text{for } \zeta \in W'. \ \mathbf{Q}$$

(3b) For all open subsets O of U_2, all $z \in W_2$ and $\varepsilon > 0$ there is a neighbourhood $W' \subset W_2$ of z with

$$\mu(\varphi^z(O)) \leqslant \mu(\varphi^\zeta(O)) + \varepsilon \quad \text{for } \zeta \in W'.$$

P Since $\varphi^z(O) \subset V_1$ for $z \in W_2$, we have $\mu(\varphi^z(O)) < \infty$. Hence there is a compact subset C of $\varphi^z(O)$ with

$$\mu(\varphi^z(O)) \leqslant \mu(C) + \varepsilon.$$

From $\psi^z \circ \varphi^z = \mathrm{id}$ on U_2 for $z \in W_2$ we deduce that $\psi^z(C) \subset O$, and hence there is, by the continuity of ψ, a neighbourhood $W' \subset W_2$ of z with

$$\psi^\zeta(C) \subset O \subset U_2 \quad \text{for } \zeta \in W'.$$

This implies $C \subset \varphi^\zeta(\psi^\zeta(C)) \subset \varphi^\zeta(O)$, whence

$$\mu(\varphi^\zeta(O)) \geqslant \mu(C) \geqslant \mu(\varphi^z(O)) - \varepsilon \quad \text{for } \zeta \in W'. \ \mathbf{Q}$$

(3c) For all compact subsets K of U_2 and all $z \in W_2$ we have

$$\mu(\varphi^z(K)) \geqslant \int_K \gamma^z \, d\lambda.$$

P Let $\varepsilon > 0$ and $z \in W_2$. By the continuity of γ and the compactness of K there is a neighbourhood $W' \subset W_2$ of z with

$$|\gamma(x, z) - \gamma(x, \zeta)| < \varepsilon \quad \text{for } \zeta \in W' \text{ and } x \in K.$$

Hence

$$\int_K \gamma^\zeta d\lambda \geqslant \int_K \gamma^z d\lambda - \varepsilon \cdot \lambda(K) \quad \text{for } \zeta \in W'.$$

Now, by (3a) there is an open neighbourhood $W'' \subset W'$ of z with

$$\mu(\varphi^z(K)) \geqslant \mu(\varphi^\zeta(K)) - \varepsilon \quad \text{for } \zeta \in W''$$

and, since $\nu(W'') > 0$ by (iii), there is by (2) a point $\zeta \in W''$ with

$$\mu(\varphi^\zeta(K)) = \int_K \gamma^\zeta d\lambda.$$

Altogether, using this ζ, we now have

$$\mu(\varphi^z(K)) \geqslant \int_K \gamma^z d\lambda - \varepsilon \cdot (1 + \lambda(K)) \quad \text{for any } \varepsilon > 0. \text{ **Q**}$$

(3d) For all open subsets O of U_2 and all $z \in W_2$ we have

$$\mu(\varphi^z(O)) \leqslant \int_O \gamma^z d\lambda.$$

P Let $\varepsilon > 0$, K be a compact subset of O with $\lambda(O \backslash K) < \varepsilon$ and $z \in W_2$. As in (3c) there is a neighbourhood $W' \subset W_2$ of z with

$$\int_K \gamma^\zeta d\lambda \leqslant \int_K \gamma^z d\lambda + \varepsilon \cdot \lambda(O) \quad \text{for } \zeta \in W',$$

and by (3b) there is an open neighbourhood $W'' \subset W'$ of z with

$$\mu(\varphi^z(O)) \leqslant \mu(\varphi^\zeta(O)) + \varepsilon \quad \text{for } \zeta \in W''.$$

Since $\nu(W'') > 0$, there is by (2) a point $\zeta \in W''$ with

$$\mu(\varphi^\zeta(O)) = \int_O \gamma^\zeta d\lambda.$$

From $\gamma(x, z) \leqslant M/\delta$ for $x \in U_2$ and $z \in W_2$ we obtain

$$\int_O \gamma^\zeta d\lambda \leqslant \int_K \gamma^\zeta d\lambda + \frac{M}{\delta} \cdot \varepsilon.$$

Altogether, we now have

$$\mu(\varphi^z(O)) \leqslant \int_O \gamma^z d\lambda + \varepsilon \cdot (1 + M/\delta + \lambda(O))$$

for any $\varepsilon > 0$. **Q**

(3e) We can now prove (3), and thus the theorem: Consider a Borel subset A of U_2. Then, for compact sets K and open sets O with $K \subset A \subset O \subset U_2$, by

(3c) and (3d) we have

$$\int_K \gamma^z d\lambda \leqslant \mu(\varphi^z(K)) \leqslant \mu(\varphi^z(A)) \leqslant \mu(\varphi^z(O)) \leqslant \int_O \gamma^z d\lambda \quad \text{for } z \in W_2.$$

The regularity of λ now implies

$$\mu(\varphi^z(A)) = \int_A \gamma^z d\lambda \qquad \text{for all } z \in W_2.$$

Remarks. (a) Theorem 1 generalizes and sharpens Kuczma's Proposition 2: all measurability assumptions on f, φ, ψ, α and β and the σ-finiteness of m can be dropped.

(b) The arguments (3b) and (3d) are an alternative to Kuczma's argument in [5], pp. 101 and 102.

3. An extension of the Steinhaus–Weil theorem. Using a method due to Weil [10], we can now derive a Steinhaus type theorem that contains Weil's and Kuczma's theorem.

THEOREM 2. *Let (X, λ), (Y, μ) and (Z, ν) be Radon measure spaces, D an open subset of $X \times Y$, and $f\colon D \to Z$ a continuous mapping. Let $A \subset X$ and $B \subset Y$ be sets of finite positive measure with $A \times B \subset D$. If f is continuously solvable and continuously RN-differentiable with non-vanishing partial RN-derivatives at every $(x, y) \in A \times B$, then $f(A \times B)$ has non-empty interior.*

Proof. By regularity we can assume A and B to be compact sets. A simple compactness argument shows that there are points $x_0 \in A$ and $y_0 \in B$ with

$$\lambda(A \cap U) > 0 \quad \text{and} \quad \mu(B \cap V) > 0$$

for all open neighbourhoods U of x_0 and V of y_0. Put $z_0 = f(x_0, y_0)$ and let U, V and W be open neighbourhoods of x_0, y_0 and z_0, respectively, such that

$$f(x_0, \varphi(x_0, z)) = z \qquad \text{for } z \in W, \; \varphi_{x_0}(W) \subset V$$

and the partial RN-derivative α of f is defined on $U \times V$. Let $W' \subset W$ be open and $z \in W'$. Then there is, by the continuity of f, an open neighbourhood $U' \subset U$ of x_0 with $f^y(U') \subset W'$, where $y = \varphi(x_0, z) \in V$. Hence

$$\nu(W') \geqslant \nu(f^y(A \cap U')) = \int_{A \cap U'} \alpha^y d\lambda > 0.$$

Thus all three conditions of Theorem 1 are satisfied at (x_0, y_0, z_0), whence there are open neighbourhoods U_0 of x_0, V_0 of y_0 and W_0 of z_0 such that $\psi^z(E)$ is measurable and

$$\lambda(\psi^z(E)) = \int_E \frac{\beta(\psi(y, z), y)}{\alpha(\psi(y, z), y)} d\mu(y)$$

for all Borel subsets E of V_0 and all $z \in W_0$. Moreover, we may assume $\nu(W_0) < \infty$, $f(U_0 \times V_0) \subset W_0$ and

$$f(x, y) = z \Leftrightarrow \psi^z(y) = x \text{ for all } (x, y, z) \in U_0 \times V_0 \times W_0.$$

Let A' and B' be compact subsets of $A \cap U_0$ and $B \cap V_0$, respectively, of positive measure, and define the function ω: $W_0 \to \boldsymbol{R}_0^+$ by

$$\omega(z) = \lambda(A' \cap \psi^z(B')).$$

We show that ω is continuous. Let $\varepsilon > 0$ and $z \in W_0$. Choose an open subset O of X with

$$\psi^z(B') \subset O \quad \text{and} \quad \lambda(O \backslash \psi^z(B')) < \varepsilon.$$

By the continuity of ψ there is a neighbourhood $W' \subset W_0$ of z with $\psi^\zeta(B') \subset O$ for $\zeta \in W'$. Hence

$$\begin{aligned} |\omega(z) - \omega(\zeta)| &\leqslant \lambda(O \backslash \psi^z(B')) + \lambda(O \backslash \psi^\zeta(B')) \quad \text{(draw a diagram!)} \\ &\leqslant 2\varepsilon + (\lambda(\psi^z(B')) - \lambda(\psi^\zeta(B'))) \end{aligned}$$

for $\zeta \in W'$. Since α, β and ψ are continuous and B' is compact, the integral representation of $\lambda(\psi^\zeta(B'))$ for $\zeta \in W_0$ now shows that ω is continuous in W_0.

Let $\hat{\lambda}$ and $\hat{\nu}$ be the completions of λ and ν. Since $x \in \psi^z(B')$ is equivalent to $z \in f_x(B')$, we obtain, using Theorem A,

$$\begin{aligned} \int_{W_0} \omega d\nu &= \int_{W_0} \Big(\int_{A'} \chi_{f_x(B')}(z) d\lambda(x) \Big) d\nu(z) = \int_{A'} \Big(\int_{W_0} \chi_{f_x(B')}(z) d\hat{\nu}(z) \Big) d\hat{\lambda}(x) \\ &= \int_{A'} \nu(f_x(B')) d\hat{\lambda}(x) = \int_{A'} \Big(\int_{B'} \beta_x d\mu \Big) d\hat{\lambda}(x) = \int_{A' \times B'} \beta d(\hat{\lambda} \times \hat{\mu}) > 0. \end{aligned}$$

Thus, $\omega(z) > 0$ in some non-empty open subset $\tilde{W}$ of W_0. But if $\omega(z) \neq 0$, then there are points $x \in A' \subset A$ and $y \in B' \subset B$ with $x = \psi^z(y)$, i.e., $f(x, y) = z$. Hence $\tilde{W} \subset f(A \times B)$, which proves the theorem.

Remarks. (a) Theorem 2 generalizes and sharpens Kuczma's theorem: all measurability assumptions on f, φ, ψ, α and β can be dropped. Moreover, the σ-finiteness of m is only needed to ensure that sets A and B of positive but possibly infinite measure, as considered by Kuczma, contain sets of finite positive measure.

(b) Theorem 2 comprises the Steinhaus–Weil theorem: If $X = Y = Z$ is a locally compact topological group and $\lambda = \mu = \nu$ a Haar measure, then the mapping $f(x, y) = x \cdot y$ for x, $y \in X$ is continuously solvable and continuously RN-differentiable at every $(x, y) \in X \times X$. The RN-derivatives are $\alpha(x, y) = \Delta(y)$ and $\beta(x, y) = 1$ for x, $y \in X$ in the case of a left Haar measure, and $\alpha(x, y) = 1$ and $\beta(x, y) = 1/\Delta(x)$ for x, $y \in X$ in the case of a right Haar measure. Here, Δ is the positive and continuous modular function of X (cf. [3], pp. 195 and 196).

(c) Of course, Kuczma's Corollary, the n-dimensional version of the theorem of Erdös and Oxtoby [2], is now a direct consequence of Theorem 2.

(d) Combining the arguments used here with those of Beck et al. [1] one can show that Theorem 2 also holds for every subset B of Y that has finite positive *outer* measure, i.e., if

$$0 < \inf\{\mu(E)\colon E \supset B, E \subset Y \text{ measurable}\} < \infty.$$

Finally, we want to draw attention to the various interesting problems posed by Kuczma in [5], pp. 105 and 106.

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