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A CONNECTED LOCALLY CONNECTED COUNTABLE SPACE WHICH IS ALMOST REGULAR

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1. Introduction. In [5] Urysohn proves that a countable connected space cannot be regular (cf. [3]). This poses the questions raised in [4] and [2] as to how closely a connected or locally connected countable space may resemble a regular space. In particular, can there be countable connected and locally connected almost regular spaces? We answer this question in the affirmative by constructing an example of such a space. The example we provide is obtained by combining the topologies of the Urysohn spaces constructed by Jones and Stone in [1] and Ritter in [3].

2. Definitions. A point p in a topological space X is called a *regular point* of X if given any open set U containing p there is an open set V containing p such that $\mathrm{Cl}(V) \subset U$. A topological space is called a *Urysohn space* if for each pair of distinct points p and q there are open sets U and V, with disjoint closures, containing p and q, respectively. An *almost regular space* is a Urysohn space which contains a dense subset of regular points.

3. A countable connected locally connected almost regular Urysohn space. Let Q_n be a countable set of rational numbers such that, for each integer n, Q_n is dense in the reals and $\{Q_n\}_{n=-\infty}^{\infty}$ is pairwise disjoint. Let $L_n \subset E^2$ be defined by $L_n = \{(x, y) \mid x \in Q_n, y = n\}$. Similarly, let $\{I_n\}_{n=-\infty}^{\infty}$ be a countable collection of pairwise disjoint sets of irrational numbers each of which is dense in the reals. Set $K_n = \{(x, y) \mid x \in I_n, y = n\}$, $L'_n = L_n \cup K_n$ and let $\{r_i\}_{i=1}^{\infty}$ and $\{w_i\}_{i=1}^{\infty}$ denote the elements of $\bigcup_{n=-\infty}^{\infty} L_n$ and $\bigcup_{n=-\infty}^{\infty} K_n$, respectively. Henceforth, to simplify the notation we shall call the r_i's rational points and the w_i's irrational points. Let

$$X_0 = \bigcup_{n=-\infty}^{\infty} L'_n \quad \text{and} \quad X_1 = \bigcup Y_n^m,$$

where each Y_n^m denotes a copy of X_0 and the union ranges over all pairs

(m, n) of positive integers. Set $Z_1 = X_0 \cup X_1$ and regard Z_1 and X_1 as disjoint unions. Thus, if $p \in Z_1$, then either $p \in X_0$ or $p \in Y_n^m$ for some unique pair (m, n) of positive integers. The basic open sets for Z_1 are defined as follows.

If $p \in L'_{2k} \subset Y_n^m$ and $\varepsilon > 0$, let

$$N_{1(1)}^{\varepsilon}(p) = \{q \in L'_{2k} \mid d(p, q) < \varepsilon\}.$$

If $p = (x, 2k+1) \in L'_{2k+1} \subset Y_n^m$ and $\varepsilon > 0$, let

$$N_{1(1)}^{\varepsilon}(p) = \{q \in L'_{2k+j} \mid j = 0, 1, 2 \text{ and } d(p_j, q) < \varepsilon, \text{ where } p_j = (x, 2k+j)\},$$

where L'_{2k} and L'_{2k+2} are the "lines" in Y_n^m one unit below and above L'_{2k+1}, respectively. The sets $N_{1(1)}^{\varepsilon}(p)$ form a basis for a topology on X_1 which satisfies the Urysohn separation property.

For each $w_i \in X_0$ and $\varepsilon > 0$, let

$$O_1^{\varepsilon}(w_i) = \{q \in L'_k \subset Y_n^i \mid n = 1, 2, \ldots \text{ and } k \geqslant 2[1/\varepsilon]\},$$

where $[1/\varepsilon]$ denotes the greatest integer less than or equal to $1/\varepsilon$. For each $r_i \in X_0$ and $\varepsilon > 0$, let

$$O_1^{\varepsilon}(r_i) = \{q \in L'_k \subset Y_i^m \mid m = 1, 2, \ldots \text{ and } k \leqslant -2[1/\varepsilon]\}.$$

If $r_i \in L'_{2k} \subset X_0$ and $\varepsilon > 0$, let

$$U_0^{\varepsilon}(r_i) = \{q \in L'_{2k} \mid d(r_i, q) < \varepsilon\}.$$

If $r_i = (x, 2k+1) \in L'_{2k+1} \subset X_0$, let

$$U_0^{\varepsilon}(r_i) = \{q \in L'_{2k+j} \subset X_0 \mid j = 0, 1, 2 \text{ and } d(p_j, q) < \varepsilon, \text{ where } p_j = (x, 2k+j)\}.$$

The basic open sets for points in X_0 are defined as follows:

$$N_{0(1)}^{\varepsilon}(w_i) = \{w_i\} \cup O_1^{\varepsilon}(w_i) \quad \text{and} \quad N_{0(1)}^{\varepsilon}(r_i) = U_0^{\varepsilon}(r_i) \cup \left(\bigcup O_1^{\varepsilon}(p)\right) \cup \left(\bigcup Y_n^m\right),$$

where $\bigcup O_1^{\varepsilon}(p)$ ranges over all $p \in U_0^{\varepsilon}(r_i)$ and $\bigcup Y_n^m$ ranges over all pairs (m, n) such that w_m and r_n are elements of $U_0^{\varepsilon}(r_i)$.

Since the following properties are easy to verify, we shall omit their formal proofs.

(i) *The collection* $\{N_{i(1)}^{\varepsilon}(p) \mid p \in Z_1, i = 0, 1$ *and* $\varepsilon > 0\}$ *forms a basis for a topology on* Z_1 *which satisfies the Urysohn separation property.*

(ii) Z_1 *is connected.*

(iii) *The elements of* X_0 *have connected* ε*-neighborhoods.*

(iv) $R_0 = \{w_i \in X_0 \mid i = 1, 2, \ldots\}$ *is the set of all regular points of* Z_1.

We note that (ii) follows from the observation that any set which is both open and closed and contains some w_i must contain Y_n^i for every positive integer n, and hence must contain r_n for every positive integer n.

To see (iii) note that an ε-neighborhood of an irrational point in X_0 is a slight modification of Roy's lattice space [6]. As for the rational points, any subset U of an ε-neighborhood of $r_n \in X_0$ which is both open and closed in U contains some w_k. Thus $Y_n^k \subset U$, which implies $r_n \in U$.

In order to prove (iv), simply observe that for each w_i in R_0 and for each sufficiently small $\varepsilon > 0$

$$N_{0(1)}^{\varepsilon/2}(w_i) \subset \mathrm{Cl}\big(N_{0(1)}^{\varepsilon/2}(w_i)\big) \subset N_{0(1)}^{\varepsilon}(w_i).$$

On the other hand, if $p \in Z_1 - R_0$, then for every pair $\varepsilon, \varepsilon' > 0$

$$\mathrm{Cl}\big(N_{i(1)}^{\varepsilon'}(p)\big) \not\subset N_{i(1)}^{\varepsilon}(p), \qquad \text{where } i = 0, 1.$$

We next extend our space so that the elements of X_1 have connected ε-neighborhoods and X_1 contains a dense subset of regular points.

With each Y_n^m we associate a countable collection $\{Y_{n,l}^{m,k}\}_{k,l=1}^{\infty}$ of copies of X_0. Let $X_2 = \bigcup Y_{n,l}^{m,k}$, where the union ranges over all 4-tuples (k, l, m, n) of positive integers, and let

$$Z_2 = \bigcup_{i=0}^{2} X_i.$$

Again regard X_i, $i = 1, 2$, and Z_2 as disjoint unions. Basic open sets for Z_2 are defined as follows.

If $p \in L'_{2k} \subset Y_{n,l}^{m,h}$ and $\varepsilon > 0$, let

$$N_{2(2)}^{\varepsilon}(p) = \{q \in L'_{2k} \mid d(p, q) < \varepsilon\}.$$

If $p = (x, 2k+1) \in L'_{2k+1} \subset Y_{n,l}^{m,h}$ and $\varepsilon > 0$, let

$$N_{2(2)}^{\varepsilon}(p) = \{q \in L'_{2k+j} \subset Y_{n,l}^{m,h} \mid j = 0, 1, 2 \text{ and } d(p_j, q) < \varepsilon, \quad \text{where } p_j = (x, 2k+j)\}.$$

This defines basic open sets for points in X_2 which satisfy the Urysohn separation property.

For each $w_j \in Y_n^m$ and $\varepsilon > 0$, let

$$O_2^{\varepsilon}(w_j) = \{q \in L'_k \subset Y_{n,l}^{m,j} \mid l = 1, 2, \ldots \text{ and } k \geqslant 2[1/\varepsilon]\}.$$

For each $r_j \in Y_n^m$ and $\varepsilon > 0$, let

$$O_2^{\varepsilon}(r_j) = \{q \in L'_k \subset Y_{n,j}^{m,l} \mid l = 1, 2, \ldots \text{ and } k \leqslant -2[1/\varepsilon]\}.$$

We now redefine the neighborhoods of points in $X_0 \cup X_1$.

For each $w_j \in Y_n^m$ and $\varepsilon > 0$, let

$$N_{1(2)}^{\varepsilon}(w_j) = N_{1(1)}^{\varepsilon}(w_j) \cup \big(\bigcup O_2^{\varepsilon}(p)\big) \cup \big(\bigcup Y_{n,l}^{m,k}\big),$$

where $\bigcup O_2^\varepsilon(p)$ ranges over all $p \in N_{1(1)}^\varepsilon(w_j)$ and $\bigcup Y_{n,l}^{m,k}$ ranges over all pairs (k, l) such that w_k and r_l are elements of $N_{1(1)}^\varepsilon(w_j)$.

For each $r_j \in Y_n^m$ and $\varepsilon > 0$, let

$$N_{1(2)}^\varepsilon(r_j) = \{r_j\} \cup O_2^\varepsilon(r_j).$$

Points $p \in X_0$ have basic ε-neighborhoods of the form

$$N_{0(2)}^\varepsilon(p) = N_{0(1)}^\varepsilon(p) \cup \left(\bigcup O_2^\varepsilon(q)\right) \cup \left(\bigcup Y_{n,l}^{m,k}\right),$$

where $\bigcup O_2^\varepsilon(q)$ ranges over all $q \in X_1 \cap N_{0(1)}^\varepsilon(p)$, and $\bigcup Y_{n,l}^{m,k}$ ranges over all 4-tuples (m, k, n, l) for which the pair w_k, r_l is in $Y_n^m \cap N_{0(1)}^\varepsilon(p)$.

The proof of the following properties is analogous to the verification of these properties for the space Z_1.

(i) *The collection* $\{N_{i(2)}^\varepsilon(p) \mid p \in Z_2, i = 0, 1, 2$ *and* $\varepsilon > 0\}$ *forms a basis for a topology on* Z_2 *which satisfies the Urysohn separation property.*

(ii) Z_2 *is connected.*

(iii) *The elements of* $X_0 \cup X_1$ *have connected* ε*-neighborhoods.*

(iv) *If* R_1 *is the set of all rational points of* X_1, *then* $R_0 \cup R_1$ *is the set of regular points of* Z_2.

Our construction of $N_{i(2)}^\varepsilon(p)$ also shows that

(v) *If* $N_{i(1)}^\varepsilon(p)$ *and* $N_{j(1)}^\varepsilon(q)$ *have disjoint closures, then so do* $N_{i(2)}^\varepsilon(p)$ *and* $N_{j(2)}^\varepsilon(q)$.

In order to obtain an almost regular space with the desired properties, we simply repeat the above process ad infinitum. The inductive argument should now be apparent. Suppose we have defined the connected Urysohn space

$$Z_k = \bigcup_{i=0}^{k} X_i$$

such that

(a) for each $i = 1, 2, \ldots, k$, $X_i = \bigcup Y_{n(1),\ldots,n(i)}^{m(1),\ldots,m(i)}$, where $m(j), n(j)$, $1 \leqslant j \leqslant i$, range over all positive integers;

(b) the elements of $\bigcup_{i=1}^{k-1} X_i$ have connected ε-neighborhoods;

(c) $\bigcup_{i=1}^{k-1} R_i$ is the set of regular points of Z_k, where R_i consists of all rational points of X_i if i is odd and R_i consists of all irrational points of X_i if i is even;

(d) if $N_{i(n)}^\varepsilon(p)$ and $N_{j(m)}^\varepsilon(q)$ have disjoint closures for $n, m < k$, then so do $N_{i(k)}^\varepsilon(p)$ and $N_{j(k)}^\varepsilon(q)$.

Then

$$X_{k+1} = \bigcup Y_{n(1),\ldots,n(k),n(k+1)}^{m(1),\ldots,m(k),m(k+1)},$$

where $\{Y^{m(1),\ldots,m(k),m(k+1)}_{n(1),\ldots,n(k),n(k+1)} \mid m(k+1)$ and $n(k+1)$ range over all positive integers$\}$ is the countable collection of copies of X_0 associated with $Y^{m(1),\ldots,m(k)}_{n(1),\ldots,n(k)}$, and

$$Z_{k+1} = \bigcup_{i=0}^{k+1} X_i.$$

Again regard X_i, $i > 0$, and Z_{k+1} as disjoint unions. Basic open sets for Z_{k+1} are defined as follows.

Let $Y = Y^{m(1),\ldots,m(k+1)}_{n(1),\ldots,n(k+1)}$. If $p \in L'_{2r} \subset Y$ and $\varepsilon > 0$, let

$$N^{\varepsilon}_{k+1(k+1)}(p) = \{q \in L'_{2r} \mid d(p, q) < \varepsilon\}.$$

If $p = (x, 2r+1) \in L'_{2r+1} \subset Y$ and $\varepsilon > 0$, let

$$N^{\varepsilon}_{k+1(k+1)}(p) = \{q \in L'_{2r+j} \subset Y \mid j = 0, 1, 2 \text{ and } d(p_j, q) < \varepsilon, \text{ where } p_j = (x, 2r+j)\}.$$

This defines the open sets for points in X_{k+1}. For each $w_j \in Y^{m(1),\ldots,m(k)}_{n(1),\ldots,n(k)}$ and $\varepsilon > 0$, let

$$O^{\varepsilon}_{k+1}(w_j) = \{q \in L'_r \subset Y^{m(1),\ldots,m(k),j}_{n(1),\ldots,n(k),l} \mid l = 1, 2, \ldots \text{ and } r \geqslant 2[1/\varepsilon]\}.$$

For each $r_j \in Y^{m(1),\ldots,m(k)}_{n(1),\ldots,n(k)}$ and $\varepsilon > 0$, let

$$O^{\varepsilon}_{k+1}(r_j) = \{q \in L'_r \subset Y^{m(1),\ldots,m(k),l}_{n(1),\ldots,n(k),j} \mid l = 1, 2, \ldots \text{ and } r \leqslant -2[1/\varepsilon]\}.$$

Define ε-neighborhoods for $w_j, r_j \in Y^{m(1),\ldots,m(k)}_{n(1),\ldots,n(k)}$ as follows.

If k is even, let

$$N^{\varepsilon}_{k(k+1)}(w_j) = \{w_j\} \cup O^{\varepsilon}_{k+1}(w_j)$$

and

$$N^{\varepsilon}_{k(k+1)}(r_j) = N^{\varepsilon}_{k(k)}(r_j) \cup \left(\bigcup O^{\varepsilon}_{k+1}(p)\right) \cup \left(\bigcup Y^{m(1),\ldots,m(k),m(k+1)}_{n(1),\ldots,n(k),n(k+1)}\right),$$

where $\bigcup O^{\varepsilon}_{k+1}(p)$ ranges over all $p \in N^{\varepsilon}_{k(k)}(r_j)$ and $\bigcup Y^{m(1),\ldots,m(k),m(k+1)}_{n(1),\ldots,n(k),n(k+1)}$ ranges over all pairs $(m(k+1), n(k+1))$ such that $w_{m(k+1)}$ and $r_{n(k+1)}$ are elements of $N^{\varepsilon}_{k(k)}(r_j)$.

If k is odd, let

$$N^{\varepsilon}_{k(k+1)}(r_j) = \{r_j\} \cup O^{\varepsilon}_{k+1}(r_j)$$

and

$$N^{\varepsilon}_{k(k+1)}(w_j) = N^{\varepsilon}_{k(k)}(w_j) \cup \left(\bigcup O^{\varepsilon}_{k+1}(p)\right) \cup \left(\bigcup Y^{m(1),\ldots,m(k),m(k+1)}_{n(1),\ldots,n(k),n(k+1)}\right),$$

where $O^{\varepsilon}_{k+1}(p)$ ranges over all $p \in N^{\varepsilon}_{k(k)}(w_j)$ and $\bigcup Y^{m(1),\ldots,m(k),m(k+1)}_{n(1),\ldots,n(k),n(k+1)}$ ranges over all pairs $(m(k+1), n(k+1))$ such that $w_{m(k+1)}$ and $r_{n(k+1)}$ are elements of $N^{\varepsilon}_{k(k)}(r_j)$.

If $p \in X_i$, $0 \leqslant i < k$, and $\varepsilon > 0$, let

$$N^{\varepsilon}_{i(k+1)}(p) = N^{\varepsilon}_{i(k)}(p) \cup \left(\bigcup O^{\varepsilon}_{k+1}(q)\right) \cup \left(\bigcup Y^{m(1),\ldots,m(k+1)}_{n(1),\ldots,n(k+1)}\right),$$

where $\bigcup O^{\varepsilon}_{k+1}(q)$ ranges over all $q \in X_k \cap N^{\varepsilon}_{i(k)}(p)$, and $\bigcup Y^{m(1),\ldots,m(k+1)}_{n(1),\ldots,n(k+1)}$ ranges over all $(2k+1)$-tuples $\big(m(1),\ldots,m(k+1),n(1),\ldots,n(k+1)\big)$ for which the pair $w_{m(k+1)}, r_{n(k+1)}$ is in $Y^{m(1),\ldots,m(k)}_{n(1),\ldots,n(k)} \cap N^{\varepsilon}_{i(k)}(p)$.

The proofs of the following properties are as before.

(i) *The collection* $\{N^{\varepsilon}_{i(k+1)}(p) \mid p \in Z_{k+1},\ i = 0, 1, \ldots, k+1$ *and* $\varepsilon > 0\}$ *forms a basis for a topology on* Z_{k+1} *which satisfies the Urysohn separation property.*

(ii) Z_{k+1} *is connected.*

(iii) *The elements of* $\bigcup_{i=0}^{k} X_i$ *have connected* ε-*neighborhoods.*

(iv) $\bigcup_{i=0}^{k} R_i$ *is the set of regular points of* Z_{k+1}.

(v) *If* $N^{\varepsilon}_{i(n)}(p)$ *and* $N^{\varepsilon}_{j(m)}(q)$ *have disjoint closures for* $n, m < k$, *then so do* $N^{\varepsilon}_{i(k+1)}(p)$ *and* $N^{\varepsilon}_{j(k+1)}(q)$.

This completes the inductive step.

Now let

$$X = \bigcup_{i=0}^{\infty} X_i.$$

Clearly, X is countable. If $p \in X$, then $p \in X_i$ for some unique integer i. For each $\varepsilon > 0$, we define $N_{\varepsilon}(p)$ to be the smallest subset of X which contains $N^{\varepsilon}_{i(k)}(p)$ for each integer $k > i$. It follows from our inductive construction that the sets $N_{\varepsilon}(p)$ form a basis for X. Since each Z_k is connected, X is connected. Furthermore, the set

$$\bigcup_{i=0}^{\infty} R_i = R$$

is dense in X.

If p and q are distinct points in X, then $p \in X_i$ and $q \in X_j$ with i and j not necessarily distinct. Let $k \geqslant \max\{i+1, j+1\}$. Since Z_k is a Urysohn space, there exists an $\varepsilon > 0$ such that $N^{\varepsilon}_{i(k)}(p)$ and $N^{\varepsilon}_{j(k)}(q)$ have disjoint closures. Thus, by property (iv), $N^{\varepsilon}_{i(n)}(p)$ and $N^{\varepsilon}_{j(n)}(q)$ have disjoint closures for every integer $n \geqslant k$. It follows that

$$\mathrm{Cl}\big(N_{\varepsilon}(p)\big) \cap \mathrm{Cl}\big(N_{\varepsilon}(q)\big) = \emptyset.$$

Hence X is a Urysohn space.

It remains to show that every point in R is regular and that X is locally connected. If $p \in R$, then $p \in R_i$ for some unique integer i. Thus for $k > i+1$ and $\varepsilon > 0$ there is a positive number $\varepsilon' > 0$ such that $\mathrm{Cl}\big(N^{\varepsilon'}_{i(k)}(p)\big) \subset [N^{\varepsilon}_{i(k)}(p)]$. But then $\mathrm{Cl}\big(N^{\varepsilon'}_{i(n)}(p)\big) \subset N^{\varepsilon}_{i(n)}(p)$ for every integer $n \geqslant k$ and, therefore, $\mathrm{Cl}\big(N_{\varepsilon'}(p)\big) \subset N_{\varepsilon}(p)$. Hence p is a regular point of X. The local connectivity follows from the fact that if $p \in X$, then $p \in X_i$ for some unique integer i. Thus, by property (iii), p has connected ε-neighborhoods in Z_k for every $k \geqslant i+1$.

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