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# ON WEAK AUTOMORPHISMS OF ALGEBRAS HAVING A BASIS

BY

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**1.** Let $A$ be a non-empty set. We define $O^{(n)} = O^{(n)}(A) = \{f \mid f\colon A^n \to A\}$ and $O = \bigcup_{n=0}^{\infty} O^{(n)}$. Given a permutation $\sigma \in S_A$, we define permutation $\sigma^*$ of $O$ by

$$(\sigma^* f)(x_1, \ldots, x_n) = \sigma f(\sigma^{-1}x_1, \ldots, \sigma^{-1}x_n). \tag{1}$$

Following Goetz (see [3]) and Marczewski we say that a permutation $\sigma$ of $A$ is a *weak automorphism* of an algebra $\mathfrak{A} = \langle A; F\rangle$ ($F \subset O$) if, for every $n$, the permutation $\sigma^*$ maps the set $A^{(n)}$ of all algebraic operations of $n$ variables (cf. [4]) of algebra $\mathfrak{A}$ onto itself.

Observe that for each weak automorphism $\sigma$ of $\mathfrak{A}$ we have

$$\sigma^* e_j^{(n)} = e_j^{(n)} \tag{2}$$

for every trivial operation $e_j^{(n)}(x_1, \ldots, x_n) = x_j$, and

$$\sigma^*\big(f(g_1, \ldots, g_n)\big) = (\sigma^* f)(\sigma^* g_1, \ldots, \sigma^* g_n) \tag{3}$$

for all $f \in A^{(n)}$, $g_1, \ldots, g_n \in A^{(n)}$.

A permutation $\varphi$ of the set $A = \bigcup_{n=0}^{\infty} A^{(n)}$, which satisfies conditions (2) and (3) together with the condition $\varphi(A^{(n)}) = A^{(n)}$ for $n = 0, 1, \ldots$, is called a *clone automorphism* of $\mathfrak{A}$ (for details see [1], p. 127). Clearly, the set $\mathrm{Aut}(A)$ of clone automorphisms forms a group.

Investigations of connections between the group of automorphisms $\mathrm{Aut}(\mathfrak{A})$ and the group of weak automorphisms $\mathrm{Aut}^*(\mathfrak{A})$ of the algebra $\mathfrak{A}$ were initiated in [2] and [5]. In particular, if $\mathfrak{A}$ has a finite independent set of generators, i.e. a finite basis, then the group $\mathrm{Aut}^*(\mathfrak{A})$ is a normal product of $\mathrm{Aut}(\mathfrak{A})$ and $\mathrm{Aut}(A)$ (see [5]).

The aim of this paper is to generalize this result.

**2.** THEOREM. *If an algebra $\mathfrak{A}$ has a basis $\{b_i\}_{i \in I}$ (not necceserily finite), then*

(a) *The mapping* $\mathrm{Aut}^*(\mathfrak{A}) \ni \tau \to (a, \tau^*) \in \mathrm{Aut}(\mathfrak{A}) \times \mathrm{Aut}(\boldsymbol{A})$, *where* $ab_i = \tau b_i$, $i \in I$, *is one-to-one and onto.*

(b) *The mapping* $\mathrm{Aut}(\boldsymbol{A}) \ni \varphi \xrightarrow{h} h_\varphi \in \mathrm{Aut}(\mathrm{Aut}(\mathfrak{A}))$, *where* $h_\varphi(a)$ *is an automorphism of* $\mathfrak{A}$ *determined by the condition*

$$h_\varphi(a) b_i = (\varphi f)(b_{j_1}, \ldots, b_{j_n}) \quad \textit{provided } ab_i = f(b_{j_1}, \ldots, b_{j_n}),$$

*is a homomorphism.*

(c) *If* $\tau_1 \to (a_1, \tau_1^*)$ *and* $\tau_2 \to (a_2, \tau_2^*)$, *then* $\tau_1\tau_2 \to (a_1 h_{\tau_1^*}(a_2), \tau_1^*\tau_2^*)$.

*Thus* $\mathrm{Aut}^*(\mathfrak{A})$ *is (isomorphic to) the normal product* $\mathrm{Aut}(\mathfrak{A})\mathrm{Aut}(\boldsymbol{A})$ *determined by the homomorphism* $h$.

**3.** First of all we show that

(i) *If* $\tau \in \mathrm{Aut}^*(\mathfrak{A})$, *then* $\{\tau b_i\}_{i \in I}$ *is a basis of* $\mathfrak{A}$.

(ii) *If* $\varphi \in \mathrm{Aut}(\boldsymbol{A})$ *and* $f \in \boldsymbol{A}^{(n)}$ *and* $g \in \boldsymbol{A}^{(k)}$, *then the equality*

$$f(b_{i_1}, \ldots, b_{i_n}) = g(b_{i_{n+1}}, \ldots, b_{i_{n+k}}) \tag{4}$$

*implies the equality*

$$(\varphi f)(b_{i_1}, \ldots, b_{i_n}) = (\varphi g)(b_{i_{n+1}}, \ldots, b_{i_{n+k}}). \tag{5}$$

Proof. Statement (i) follows readily from [2]. To prove (ii) suppose that elements of the sequence $i_1, \ldots, i_{n+k}$ belong to the set $\{j_1, \ldots, j_r\}$. Then we can write (4) in the form

$$f(e_{p_1}^{(r)}, \ldots, e_{p_n}^{(r)})(b_{j_1}, \ldots, b_{j_r}) = g(e_{p_{n+1}}^{(r)}, \ldots, e_{p_{n+k}}^{(r)})(b_{j_1}, \ldots, b_{j_r})$$

for some $p_1, \ldots, p_{n+k} \in \{1, \ldots, r\}$. Hence

$$f(e_{p_1}^{(r)}, \ldots, e_{p_n}^{(r)}) = g(e_{p_{n+1}}^{(r)}, \ldots, e_{p_{n+k}}^{(r)}).$$

In view of (2) and (3) we get

$$(\varphi f)(e_{p_1}^{(r)}, \ldots, e_{p_n}^{(r)}) = (\varphi g)(e_{p_{n+1}}^{(r)}, \ldots, e_{p_{n+k}}^{(r)})$$

and (4) follows.

**4. Proof of the theorem.** To simplify notation let $\boldsymbol{b}$ (or $\boldsymbol{b}$ with an index) denote a finite sequence of the form $b_{j_1}, \ldots, b_{j_n}$.

(a) If $\tau_1 a \neq \tau_2 a$ and $a = f(\boldsymbol{b})$, then $(\tau_1^* f)(\tau_1^* \boldsymbol{b}) \neq (\tau_2^* f)(\tau_2^* \boldsymbol{b})$. Hence if $\tau_1 b_i = \tau_2 b_i$, $i \in I$, then $\tau_1^* = \tau_2^*$. Therefore the mapping $\tau \to (a, \tau^*)$ is one-to-one.

Given $(a, \varphi) \in \mathrm{Aut}(\mathfrak{A}) \times \mathrm{Aut}(\boldsymbol{A})$ and $a = f(\boldsymbol{b})$, define

$$\sigma a = (\varphi f)(a\boldsymbol{b}).$$

In view of (ii), $\sigma$ does not depend on the choice of $f$ and so it is a transformation of $A$ (if $a$ ranges over $A$). Further, if

$$\sigma a = (\varphi f)(\boldsymbol{b}_1) = (\varphi g)(\boldsymbol{b}_2) = \sigma c,$$

then, by (ii),

$$a = f(\boldsymbol{b}_1) = \big(\varphi^{-1}(\varphi f)\big)(\boldsymbol{b}_1) = \big(\varphi^{-1}(\varphi g)\big)(\boldsymbol{b}_2) = g(\boldsymbol{b}_2) = c.$$

Since $\{ab_i\}_{i\in I}$ is a basis of $\mathfrak{A}$, every element $a\in A$ is of the form $a = f(a\boldsymbol{b})$. Hence $\sigma(\varphi^{-1}f)(\boldsymbol{b}) = f(a\boldsymbol{b}) = a$ and, consequently, $\sigma$ is a permutation of $A$.

Suppose now that $a_i = g_i(\boldsymbol{b})$ for $i = 1, \ldots, m$. Then

$$\begin{aligned}\sigma f(a_1, \ldots, a_m) &= \sigma f(g_1, \ldots, g_m)(\boldsymbol{b})\\ &= (\varphi f)(g_1, \ldots, g_m)(\boldsymbol{b}) = (\varphi f)(\sigma a_1, \ldots, \sigma a_m)\end{aligned}$$

and so $\sigma$ is a weak automorphism of $\mathfrak{A}$.

(b) First we prove that

(iii) *If $a\in\mathrm{Aut}(\mathfrak{A})$, $ab_i = f_i(\boldsymbol{b}_i)$, $i\in I$, and $\varphi\in\mathrm{Aut}(\boldsymbol{A})$, then the set $\{\varphi f_i(\boldsymbol{b}_i)\}_{i\in I}$ is a basis of $\mathfrak{A}$.*

In fact, because by virtue of (a) there exists a weak automorphism $\tau$ of $\mathfrak{A}$ such that $\tau\to(a, \varphi)$, then, by (i), the image of the basis $\{f_i(b_i)\}_{i\in I}$ is a basis of $\mathfrak{A}$. On the other hand, $\tau f_i(\boldsymbol{b}_i) = (\varphi f_i)(a\boldsymbol{b}_i)$, $i\in I$. Since $(\varphi f_i)(\boldsymbol{b}_i) = a^{-1}(\varphi f_i)(a\boldsymbol{b}_i)$, the result follows.

In view of what we have already shown, the mapping $h_\varphi(a)$ maps the basis $\{b_i\}_{i\in I}$ onto a basis $\{(\varphi f_i)(\boldsymbol{b}_i)\}_{i\in I}$ (we assume $ab_i = f_i(\boldsymbol{b}_i)$). Hence $h_\varphi(a)$ is an automorphism of $\mathfrak{A}$.

Suppose now that $h_\varphi(a) = h_\varphi(\beta)$ and

$$ab_i = f_i(\boldsymbol{b}_i), \quad \beta b_i = g_i(\boldsymbol{b}i), \quad i\in I. \tag{6}$$

This yields

$$(\varphi f_i)(\boldsymbol{b}_i) = h_\varphi(a)b_i = h_\varphi(\beta)b_i = (\varphi g_i)(\boldsymbol{b}_i).$$

Because of (iii), we get

$$ab_i = f_i(\boldsymbol{b}_i) = \big(\varphi^{-1}(\varphi f_i)\big)(\boldsymbol{b}_i) = \big(\varphi^{-1}(\varphi g_i)\big)(\boldsymbol{b}_i) = g_i(\boldsymbol{b}_i) = \beta b_i, \quad i\in I,$$

which gives the equality $a = \beta$.

Further, if $a$ is an automorphism of $\mathfrak{A}$ such that $ab_i = f_i(\boldsymbol{b}_i)$, $i\in I$, then, by (iii), the set $\{(\varphi^{-1}f_i)(\boldsymbol{b}_i)\}_{i\in I}$ is a basis of $\mathfrak{A}$. Consequently, the mapping $\beta\colon b_i\to(\varphi^{-1}f_i)(\boldsymbol{b}_i)$, $i\in I$, determines an automorphism of $\mathfrak{A}$. Clearly, $h_\varphi(\beta) = a$. Hence the transformation $h_\varphi$ is a permutation of the set $\mathrm{Aut}(\mathfrak{A})$. Moreover, if $a$ and $\beta$ are automorphisms of $\mathfrak{A}$ for which (6) holds, then

$$h_\varphi(a\cdot\beta)b_i = \varphi\big(g_i(f_1, \ldots, f_m)\big)(\boldsymbol{b}),$$

$$\begin{aligned}h_\varphi(a)h_\varphi(\beta)b_i &= h_\varphi(a)g_i(\boldsymbol{b}_i) = (\varphi g_i)\big(h_\varphi(a)f_1(\boldsymbol{b}), \ldots, h_\varphi(a)f_m(\boldsymbol{b})\big)\\ &= (\varphi g_i)(\varphi f_1, \ldots, \varphi f_m)(\boldsymbol{b}),\end{aligned}$$

and therefore $h_\varphi$ is an automorphism of $\mathrm{Aut}(\mathfrak{A})$.

Let $a$ be an automorphism of $\mathfrak{A}$ with $ab_i = f_i(\boldsymbol{b}_i)$, $i \in I$. We have

$$h_{\varphi\psi}(a)b_i = \big((\varphi\psi)f_i\big)(\boldsymbol{b}_i) = \big(\varphi(\psi f_i)\big)(\boldsymbol{b}_i) = h_\varphi\big(h_\psi(\varphi)b_i\big),$$

which completes the proof of part (b).

(c) Let us suppose that $\tau_1 \to (a_1, \tau_1^*)$ and $\tau_2 \to (a_2, \tau_2^*)$. It follows from (1) that $\tau_1\tau_2 \to \big(\beta, (\tau_1 \cdot \tau_2)^*\big)$, where $\beta$ is an automorphism of $\mathfrak{A}$. Since $(\tau_1\tau_2)^* = \tau_1^*\tau_2^*$, we must only show that $\beta$ is equal to $a_1 h_{\tau_1^*}(a_2)$.

Let $a_2 b_i = f_i(\boldsymbol{b}_i)$, $i \in I$. Then

$$\beta b_i = \tau_1\tau_2 b_i = \tau_1 f_i(\boldsymbol{b}_i) = (\tau_1^* f_i)(a_1\boldsymbol{b}_i),$$

$$a_1 h_{\tau_1^*}(a_2) b_i = a_1(\tau_1^* f_i)(\boldsymbol{b}_i) = (\tau_1^* f_i)(a_1\boldsymbol{b}_i), \qquad i \in I.$$

Since $\beta$ and $a_1 h_{\tau_1^*}(a_2)$ are equal on a basis, they are equal everywhere and the theorem follows.

Remark. The converse of the theorem is false. In the algebra $\mathfrak{A} = \langle\{1, 2, 3\}; S_3\rangle$ there is $\mathrm{Aut}(\mathfrak{A}) = 1$, $\mathrm{Aut}^*(\mathfrak{A}) = S_3$, $\mathrm{Aut}(\boldsymbol{A}) = \mathrm{Aut}(S_3) = S_3$, and therefore $\mathrm{Aut}^*(\mathfrak{A})$ is the normal product of Aut $(\mathfrak{A})$ and $\mathrm{Aut}(\boldsymbol{A})$, but the algebra $\mathfrak{A}$ has no basis (1).

(1) This example was suggested to me by J. Dudek.

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