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ON EXISTENCE AND AN ASYMPTOTIC BEHAVIOR OF RANDOM SOLUTIONS OF A CLASS OF STOCHASTIC FUNCTIONAL-INTEGRAL EQUATIONS

BY

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1. Introduction. Stochastic integral equations play a significant role in the characterizing of many social, physical, biological and engineering problems. Presentation of the basic theoretical developments and an illustration of the usefulness of stochastic integral equations can be found in the monographs [3–7], [9] and [10]. This paper deals with the problem of determination of sufficient conditions which assure the existence of random solutions of a class of stochastic functional-integral equations. Moreover, we are interested in asymptotic behavior of those solutions. This problem was considered for instance in [3], [6] and [8]. The proofs of the existence theorems contained in [3] and [8] are based on the fixed point theorems of Schauder and Tychonoff.

We present here new existence theorems for a class of stochastic functional-integral equations treated in [8]. Our method is based on the notion of a measure of noncompactness in Banach spaces and the fixed point theorem of Darbo type. More precisely, we construct a real Banach space of tempered functions and next define measures of noncompactness on that space where we are seeking solutions of the considered equations. The given approach allows us also to investigate the behavior of solutions at infinity.

We shall deal with stochastic functional-integral equations of the form

$$x(t;\omega) = h(t;\omega)+\int_0^t k(t,\tau;\omega)f(\tau, x_\tau(\omega))\,d\tau, \tag{1.1}$$

$$x(t;\omega) = h(t;\omega)+\int_0^\infty k(t-\tau;\omega)f(\tau, x_\tau(\omega))\,d\tau, \tag{1.2}$$

where

(i) $t\in \boldsymbol{R}^+$ and $\omega\in\Omega$, the supporting set of a complete probability space $(\Omega, \mathfrak{A}, \mathscr{P})$,

(ii) $x(t; \omega)$ denotes an unknown random function defined for $t \in \boldsymbol{R}^+$ and $\omega \in \Omega$,

(iii) $h(t; \omega)$ is the stochastic free term defined for $t \in \boldsymbol{R}^+$ and $\omega \in \Omega$,

(iv) $k(t, \tau; \omega)$ is the stochastic kernel defined for $0 \leqslant \tau \leqslant t \leqslant \infty$ and $\omega \in \Omega$,

(v) $x_t(\omega)$ denotes the restriction of the function $x(\tau; \omega)$ to the interval $[0, t]$, $t > 0$, with $x_0(\omega) = x(0; \omega) \in L^2(\Omega, \mathfrak{A}, \mathscr{P})$.

2. Definitions and notations.

Definition 1. We shall call $x(t; \omega)$ a *random solution* to equation (1.1) if, for every fixed $t \in \boldsymbol{R}^+$, $x(t, \omega) \in L^2(\Omega, \mathfrak{A}, \mathscr{P})$ and satisfies equation (1.1) $\mathscr{P}$-a.s.

$\mathscr{X}$ will denote an infinite dimensional real Banach space with norm $\| \ \|$ and the zero element Θ. In the sequel we will denote by $K(x, r)$ the closed ball centered at x and with radius r. Denote by $\mathfrak{M}_{\mathscr{X}}$ the family of all nonempty bounded subsets of $\mathscr{X}$. Analogously, we denote by $\mathfrak{N}_{\mathscr{X}}$ the family of all relatively compact and nonempty subsets of $\mathscr{X}$.

The system of axioms defining a measure of noncompactness is taken from [2].

Definition 2. A nonempty family $\mathfrak{P} \subset \mathfrak{N}_{\mathscr{X}}$ is said to be the *kernel* (of a measure of noncompactness) provided it satisfies the following conditions:

1° $U \in \mathfrak{P} \Rightarrow \bar{U} \in \mathfrak{P}$,

2° $U \in \mathfrak{P}$, $V \subset U$, $V \neq \emptyset \Rightarrow V \in \mathfrak{P}$,

3° $U, V \in \mathfrak{P} \Rightarrow \lambda U + (1-\lambda) V \in \mathfrak{P}$ for $\lambda \in [0, 1]$,

4° $U \in \mathfrak{P} \Rightarrow \operatorname{Conv} U \in \mathfrak{P}$,

5° $\mathfrak{P}^c$ (the subfamily of $\mathfrak{P}$ consisting of all closed sets) is closed in $\mathfrak{M}_{\mathscr{X}}$ with respect to Hausdorff topology.

Definition 3. The function μ: $\mathfrak{M}_{\mathscr{X}} \to [0, +\infty)$ is said to be a *measure of noncompactness* with the kernel $\mathfrak{P}$ ($\ker \mu = \mathfrak{P}$) if it is subject to the following conditions:

1° $\mu(U) = 0 \Leftrightarrow U \in \mathfrak{P}$,

2° $\mu(U) = \mu(\bar{U})$,

3° $\mu(\operatorname{Conv} U) = \mu(U)$,

4° $U \subset V \Rightarrow \mu(U) \leqslant \mu(V)$,

5° $\mu(\lambda U + (1-\lambda) V) \leqslant \lambda \mu(U) + (1-\lambda) \mu(V)$ for $\lambda \in [0, 1]$,

6° if $U_n \in \mathfrak{M}_{\mathscr{X}}$, $U_n = \bar{U}_n$ and $U_{n+1} \subset U_n$ for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty} \mu(U_n) = 0$, then $U_\infty = \bigcap_{n=1}^{\infty} U_n \neq \emptyset$.

If a measure of noncompactness μ satisfies in addition the following two conditions:

7° $\mu(U + V) \leqslant \mu(U) + \mu(V)$,

8° $\mu(\lambda U) = |\lambda| \mu(U)$, $\lambda \in \boldsymbol{R}$,

it will be called *sublinear*.

Let $M \subset \mathscr{X}$ be a nonempty set and μ a measure of noncompactness on $\mathscr{X}$.

Definition 4. We say that a mapping $T\colon M \to \mathscr{X}$ is a *contraction* with respect to μ (μ-contraction) if $TU \in \mathfrak{M}_{\mathscr{X}}$ for any set $U \in \mathfrak{M}_{\mathscr{X}}$ and there exists a constant $k \in [0, 1)$ such that

$$\mu(TU) \leqslant k\mu(U).$$

We shall use the following modified version of the fixed point theorem of Darbo type.

THEOREM 2.1 [2]. *Let C be a nonempty, closed, convex and bounded set of $\mathscr{X}$ and let $T\colon C \to C$ be an arbitrary μ-contraction. Then T has at least one fixed point and the set* $\operatorname{Fix} T = \{x \in C\colon Tx = x\}$ *of fixed points of T belongs to* $\ker \mu$.

Further, let $L^2(\Omega, \mathfrak{A}, \mathscr{P})$ denote the space of random functions $x(t;\cdot)$ such that $|x(t;\cdot)|^2$ is integrable, with

$$\|x(t)\|_{L^2} = E^{1/2}|x(t)| = \Big(\int_{\Omega} |x(t;\omega)|^2\, dP(\omega)\Big)^{1/2}.$$

Let $p(\cdot)$ be a positive continuous function defined on $[0, +\infty)$ such that $\lim_{T\to\infty} \sup_{t \geqslant T} p(t) = 0$.

By $C_p(\boldsymbol{R}^+, L^2(\Omega, \mathfrak{A}, \mathscr{P}), p)$ (or shortly C_p) we denote a space of all continuous maps $x(t;\cdot)$ from $\boldsymbol{R}^+$ into $L^2(\Omega, \mathfrak{A}, \mathscr{P})$, with the topology defined by the norm

$$\|x\|_p = \sup[p(t)\|x(t)\|_{L^2}\colon t \geqslant 0] < \infty.$$

The space C_p with norm $\| \; \|_p$ is a real Banach space (cf. [1], [11]). Now for fixed $x \in C_p$, $U \in \mathfrak{M}_{C_p}$, $T > 0$ and $\varepsilon > 0$ we put

$$\beta^T(x, \varepsilon) = \sup\big[\|p(t)x(t) - p(s)x(s)\|_{L^2}\colon t, s \in [0, T], |t-s| \leqslant \varepsilon\big],$$

$$\beta^T(U, \varepsilon) = \sup[\beta^T(x, \varepsilon)\colon x \in U],$$

$$\beta_0^T(U) = \lim_{\varepsilon\to 0} \beta^T(U, \varepsilon),$$

$$\beta_0(U) = \lim_{T\to\infty} \beta_0^T(U),$$

$$a(U) = \lim_{T\to\infty} \sup_{x\in U}[\sup_{t\geqslant T} p(t)\|x(t)\|_{L^2}],$$

$$b(U) = \lim_{T\to\infty} \sup_{x\in U}[\sup_{t,s\geqslant T} \|p(t)x(t) - p(s)\dot{x}(s)\|_{L^2}],$$

$$\mu_0(U) = \beta_0(U) + a(U) + \sup\big[p(t)m(U(t))\colon t \geqslant 0\big],$$

$$\mu_1(U) = \beta_0(U) + b(U) + \sup\big[p(t)m(U(t))\colon t \geqslant 0\big],$$

where m is a sublinear measure of noncompactness defined on $L^2(\Omega, \mathfrak{A}, \mathscr{P})$, such that one-point sets belong to $\ker m$, and $U(t) = [x(t)\colon x \in U]$. The functions μ_0 and μ_1 define sublinear measures of noncompactness defined on C_p ([1], [2]). It is also known ([1], [2]) that $\ker \mu_0$ is the set of all sets $U \in \mathfrak{M}_{C_p}$ such that functions belonging to U are equicontinuous on any compact of $\boldsymbol{R}^+$ and $\lim_{t\to\infty} p(t)\|x(t)\|_{L^2} = 0$ uniformly with respect to $x \in U$. Further properties of μ_0 and μ_1 can be found in [1], [2].

3. **Main results.** Let $t \in \boldsymbol{R}^+$ be fixed. We assume that the stochastic kernel $k(t, \tau; \omega)$ is measurable for each t, $0 \leqslant \tau \leqslant t$, $\mathscr{P}$-essentially bounded and continuous as a map from the set $\Delta \equiv \{(t, \tau);\ 0 \leqslant \tau \leqslant t < \infty\}$ into $L_\infty(\Omega, \mathfrak{A}, \mathscr{P})$.

Define for $0 \leqslant \tau \leqslant t < \infty$,

$$|||k(t, \tau)||| = \mathscr{P}\text{-ess}\sup_{\omega\in\Omega} |k(t, \tau; \omega)|.$$

The above assumption implies that if $x \in C_p$ then for each $t \in \boldsymbol{R}^+$

$$\|k(t, \tau)x_\tau\|_{L^2} \leqslant |||k(t, \tau)|||\, \|x_\tau\|_{L^2}. \tag{3.1}$$

THEOREM 3.1. *Let the stochastic functional-integral equation* (1.1) *satisfy the following conditions*:

(i) $$|f(t, x_t(\omega))| \leqslant u(t)|x_t(\omega)| + v(t) \quad \mathscr{P}\text{-}a.s.,$$

where nonnegative functions u and v are continuous and defined for $t \in \boldsymbol{R}^+$,

(ii) *the mapping $x(t; \omega) \to f(t, x_t(\omega))$ from $C_p(\boldsymbol{R}^+, L^2(\Omega, \mathfrak{A}, \mathscr{P}), p)$ into $C_p(\boldsymbol{R}^+, L^2(\Omega, \mathfrak{A}, \mathscr{P}), p)$ is continuous in the topology generated by the norm*

$$\|x\|_p = \sup[p(t)\|x(t)\|_{L^2}\colon t \geqslant 0],$$

(iii) $$\sup\Big\{p(t)\int_0^t |||k(t, \tau)|||\big(u(\tau)/p(\tau)\big)d\tau\colon t \in \boldsymbol{R}^+\Big\} = A, \qquad 0 \leqslant A < 1,$$

(iv) $$\sup\Big\{p(t)\int_0^t |||k(t, \tau)|||\, v(\tau)d\tau\colon t \in \boldsymbol{R}^+\Big\} = B < \infty,$$

(v) $$\lim_{t\to\infty} p(t)\|h(t)\|_{L^2} = 0, \qquad \lim_{t\to\infty} p(t)\int_0^t |||k(t, \tau)|||\, v(\tau)d\tau = 0,$$

(vi) $$\lim_{t\to\infty} p(t)\|f(t, x_t) - f(t, y_t)\|_{L^2} = 0$$

uniformly with respect to x_t and y_t belonging to the ball $K(\Theta, r)$,

(vii) $$m\Big(\int_0^t k(t, \tau; \omega)f\big(\tau, U(\tau)\big)d\tau\Big) = 0 \quad \textit{for any } t \geqslant 0 \textit{ and } U \in \mathfrak{M}_{C_p}.$$

Then there exists at least one solution $x \in C_p$ *of* (1.1) *such that*

$$\lim_{t \to \infty} p(t)\,\|x(t)\|_{L^2} = 0.$$

Proof. Define the map H on C_p by

$$(Hx)(t;\omega) = h(t;\omega) + \int_0^t k(t,\tau;\omega) f\big(\tau; x(\omega)\big)\,d\tau. \tag{3.2}$$

Using the assumptions of Theorem 3.1 and (3.1) we have for $x \in C_p$

$$\begin{aligned} p(t)\,\|(Hx)(t)\|_{L^2} &\leqslant p(t)\,\|h(t)\|_{L^2} + p(t)\Big\|\int_0^t k(t,\tau) f(\tau, x_\tau)\,d\tau\Big\|_{L^2} \\ &\leqslant p(t)\,\|h(t)\|_{L^2} + p(t)\int_0^t |||k(t,\tau)|||\,\|f(\tau, x_\tau)\|_{L^2}\,d\tau \\ &\leqslant p(t)\,\|h(t)\|_{L^2} + p(t)\int_0^t |||k(t,\tau)|||\,u(\tau)\,\|x_\tau\|_{L^2}\,d\tau \\ &\quad + p(t)\int_0^t |||k(t,\tau)|||\,v(\tau)\,d\tau \\ &\leqslant \|h\|_p + \|x\|_p\,p(t)\int_0^t \big(|||k(t,\tau)|||\,u(\tau)/p(\tau)\big)\,d\tau \\ &\quad + p(t)\int_0^t |||k(t,\tau)|||\,v(\tau)\,d\tau. \end{aligned} \tag{3.3}$$

Hence, we get

$$\|Hx\|_p \leqslant \|h\|_p + A\,\|x\|_p + B. \tag{3.4}$$

Thus we have proved that H maps C_p into C_p, and moreover, we see that H maps the ball $K(\Theta, r)$ into itself with $r = (\|h\|_p + B)/(1-A)$. We now prove that the map H is continuous in the ball $K(\Theta, r)$. Let $x, y \in K(\Theta, r)$. For any given $\varepsilon_2 > 0$ choose $T > 0$ such that

$$p(\tau)\,\|f(\tau, x_\tau) - f(\tau, y_\tau)\|_{L^2} < \varepsilon_2, \quad \text{whenever } \tau > T. \tag{3.5}$$

Moreover, we can assume without loss of generality that there exists $T > 0$ such that $u(t) \geqslant 1$ whenever $t \geqslant T$, and $\min\{u(\tau)\colon 0 \leqslant \tau \leqslant T\} = u_T > 0$. Hence, putting $\max\{p(\tau)\colon 0 \leqslant \tau \leqslant T\} = p_T$, we have for $t \geqslant T$

$$p(t)\,\|(Hx)(t) - (Hy)(t)\|_{L^2} \leqslant p(t)\int_0^t |||k(t,\tau)|||\,\|f(\tau, x_\tau) - f(\tau, y_\tau)\|_{L^2}\,d\tau$$

$$\leqslant p(t)(p_T/u_T)\int_0^T (|||k(t,\tau)|||u(\tau)/p(\tau))\,p(\tau)\,\|f(\tau,x_\tau)-f(\tau,y_\tau)\|_{L^2}\,d\tau$$

$$+p(t)\int_T^t (|||k(t,\tau)|||\,u(\tau)/p(\tau))\,p(\tau)\,\|f(\tau,x_\tau)-f(\tau,y_\tau)\|_{L^2}\,d\tau,$$

which by (ii), (iii), (vi), and (3.5) implies

(3.6)
$$\sup_{t\geqslant T} p(t)\,\|(Hx)(t)-(Hy)(t)\|_{L^2}\leqslant (p_T/u_T)\,A\varepsilon_1+A\varepsilon_2, \quad \text{provided } \|x-y\|_p<\delta.$$

It can be seen that for any given $\varepsilon_3>0$ one has

(3.7)
$$\sup_{0\leqslant t\leqslant T} p(t)\,\|(Hx)(t)-(Hy)(t)\|_{L^2}<\varepsilon_3, \quad \text{whenever } \|x-y\|_p<\delta.$$

Thus, by (3.6) and (3.7), for any given $\varepsilon>0$

$$\|Hx-Hy\|_p<\varepsilon, \quad \text{whenever } \|x-y\|_p<\delta,\ x,\ y\in K(\Theta,r).$$

Let now be given $\varepsilon>0$, $T>0$ and $t,\ s\in[0,T]$, $|t-s|\leqslant\varepsilon$. By (3.2), for $0\leqslant s<t$ and $x\in U\subset K(\Theta,r)$ we have

(3.8)
$$\|(Hx)(t)\,p(t)-(Hx)(s)\,p(s)\|_{L^2}\leqslant |p(t)-p(s)|\,\|h(t)\|_{L^2}$$
$$+p(s)\,\|h(t)-h(s)\|_{L^2}+|p(t)-p(s)|\,\Big\|\int_0^t k(t,\tau)\,f(\tau,x_\tau)\,d\tau\Big\|_{L^2}$$
$$+p(s)\,\Big\|\int_0^s (k(t,\tau)-k(s,\tau))\,f(\tau,x_\tau)\,d\tau\Big\|_{L^2}+p(s)\,\Big\|\int_0^t k(t,\tau)\,f(\tau,x_\tau)\,d\tau\Big\|_{L^2}.$$

But using (3.1) with x_τ replaced by $f(\tau,x_\tau)$, we get

(3.9)
$$|p(t)-p(s)|\,\Big\|\int_0^t k(t,\tau)\,f(\tau,x_\tau)\,d\tau\Big\|_{L^2}$$
$$\leqslant |p(t)-p(s)|\int_0^t |||k(t,\tau)|||\,\|u(\tau)\,|x_\tau|+v(\tau)\|_{L^2}\,d\tau$$
$$\leqslant T\,|p(t)-p(s)|\,\big(\|x\|_p\max\{|||k(t,\tau)|||\,(u(\tau)/p(\tau)):\ 0\leqslant\tau\leqslant T\}$$
$$+\max\{|||k(t,\tau)|||\,v(\tau):\ 0\leqslant\tau\leqslant T\}\big).$$

Now we have the following estimates:

(3.10)
$$p(s)\,\Big\|\int_0^s (k(t,\tau)-k(s,\tau))\,f(\tau,x_\tau)\,d\tau\Big\|_{L^2}$$
$$\leqslant p(s)\int_0^s |||k(t,\tau)-k(s,\tau)|||\,\|u(\tau)\,|x_\tau|+v(\tau)\|_{L^2}\,d\tau$$

$$\leqslant Tp(s)r\max\{|||k(t,\tau)-k(s,\tau)|||\,(u(\tau)/p(\tau)):\ 0\leqslant\tau\leqslant T\}$$
$$+Tp(s)\max\{|||k(t,\tau)-k(s,\tau)|||\,v(\tau):\ 0\leqslant\tau\leqslant T\}$$

and

$$p(s)\Big\|\int_s^t k(t,\tau)f(\tau,x_\tau)\,d\tau\Big\|_{L^2} \tag{3.11}$$
$$\leqslant p(s)r\max\{|||k(t,\tau)|||\,(u(\tau)/p(\tau)):\ 0\leqslant\tau\leqslant T\}\,|t-s|$$
$$+p(s)\max\{|||k(t,\tau)|||\,v(\tau):\ 0\leqslant\tau\leqslant T\}\,|t-s|.$$

We now need to recall the definition of the modulus of continuity which is defined for a real function w:

$$v_T(w;\varepsilon)=\sup\{|w(t)-w(s)|:\ t,s\in[0,T],\ |t-s|\leqslant\varepsilon\},\qquad \varepsilon\geqslant 0. \tag{3.12}$$

By (3.8) and the assumptions of Theorem 3.1 we have

$$\lim_{\varepsilon\to 0}v_T(p;\varepsilon)=\lim_{\varepsilon\to 0}v_T(h;\varepsilon)=\lim_{\varepsilon\to 0}v_T(|||k|||,\varepsilon)=\lim_{\varepsilon\to 0}v_T(t,\varepsilon)=0.$$

Hence, using (3.8)–(3.11), we get

$$\beta_0((HU))=0. \tag{3.13}$$

Fix now $U\subset K(\Theta,r)$. We prove that

$$a((HU))\leqslant Aa(U). \tag{3.14}$$

It is clear, by the definition of integral, that for any given $\varepsilon_1>0$ there exists a positive integer $n_1=n_1(\varepsilon_1)$ such that for $n\geqslant n_1$

$$\Big|\int_0^t |||k(t,\tau)|||\,(u(\tau)\|x_\tau\|_{L^2}/p(\tau))\,p(\tau)\,d\tau$$
$$-\sum_{k=0}^{n-1}\frac{t}{n}\Big|\Big|\Big|k\Big(t,\frac{kt}{n}\Big)\Big|\Big|\Big|\Big(u\Big(\frac{kt}{n}\Big)\|x_{kt/n}\|_{L^2}/p\Big(\frac{kt}{n}\Big)\Big)p\Big(\frac{kt}{n}\Big)\Big|<\varepsilon_1.$$

Now let $T<t$. Put $k^*=\min[k:\ 0\leqslant k\leqslant n,\ kt/n<T]$. Then we have

$$\int_0^t |||k(t,\tau)|||\,(u(\tau)\,\|x_\tau\|_{L^2}/p(\tau))\,p(\tau)\,dt$$
$$\leqslant\varepsilon_1+\sum_{k=0}^{k^*}\frac{t}{n}\Big|\Big|\Big|k\Big(t,\frac{kt}{n}\Big)\Big|\Big|\Big|\Big(u\Big(\frac{kt}{n}\Big)\|x_{kt/n}\|_{L^2}/p\Big(\frac{kt}{n}\Big)\Big)p\Big(\frac{kt}{n}\Big)$$
$$+\sum_{k=k^*+1}^{n}\frac{t}{n}\Big|\Big|\Big|k\Big(t,\frac{kt}{n}\Big)\Big|\Big|\Big|\Big(u\Big(\frac{kt}{n}\Big)\|x_{kt/n}\|_{L^2}/p\Big(\frac{kt}{n}\Big)\Big)p\Big(\frac{kt}{n}\Big).$$

But, for any given $\varepsilon_2 > 0$

$$\sum_{k=0}^{k^*} \frac{t}{n} \left|\left|\left| k\left(t, \frac{kt}{n}\right)\right|\right|\right| \left(\left\| u\left(\frac{kt}{n}\right)\right\|_{L^2/p}\left(\frac{kt}{n}\right)\right) \|x_{kt/n}\|_{L^2} p\left(\frac{kt}{n}\right)$$

$$\leqslant k^* t \max\left[\|x_{kt/n}\|_{L^2} p\left(\frac{kt}{n}\right): \frac{kt}{n} < T\right]$$

$$\times \max\{|||k(t, \tau)|||\,(u(\tau)/p(\tau)): 0 \leqslant \tau \leqslant T\} n^{-1} < \varepsilon_2$$

for sufficiently large n. Similarly, for any $\varepsilon_3 > 0$

$$\sum_{k=k^*+1}^{n} \frac{t}{n} \left|\left|\left| k\left(t, \frac{kt}{n}\right)\right|\right|\right| \left(u\left(\frac{kt}{n}\right)/p\left(\frac{kt}{n}\right)\right) \|x_{kt/n}\|_{L^2} p\left(\frac{kt}{n}\right)$$

$$\leqslant \sup[\|x_t\|_{L^2} p(t): t \geqslant T]\left(\int_0^t |||k(t, \tau)|||\,(u(\tau)/p(\tau))\,d\tau + \varepsilon_2\right)$$

when n is sufficiently large. Therefore we have

$$p(t)\int_0^t |||k(t, \tau)|||\,(u(\tau)/p(\tau))\,\|x\|_{L^2} p(\tau)\,d\tau$$

$$\leqslant p(t)(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \sup[\|x_t\|_{L^2} p(t): t \geqslant T])$$

$$+ \sup[\|x_t\|_{L^2} p(t): t \geqslant T]\, p(t)\int_0^t |||k(t, \tau)|||\,(u(\tau)/p(\tau))\,d\tau$$

$$\leqslant p(t)[\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \|x\|_p] + A \sup[\|x_t\|_{L^2} p(t): t \geqslant T].$$

In view of the above inequalities we conclude that

$$p(t)\|(Hx)(t)\|_{L^2}$$

$$\leqslant p(t)\|h(t)\|_{L^2} + p(t)[\varepsilon_1 + \varepsilon_2 + \varepsilon_3 r] + A \sup[\|x_t\|_{L^2} p(t): t \geqslant T]$$

$$+ p(t)\int_0^t |||k(t, \tau)|||\, v(\tau)\,d\tau.$$

Thus, by the assumptions of Theorem 3.1 we obtain

$$\lim_{T\to\infty} \sup_{x\in U}[\sup[p(t)\|(Hx)(t)\|_{L^2}: t \geqslant T]]$$

$$\leqslant \lim_{T\to\infty}[\sup[p(t)\|h(t)\|_{L^2}: t \geqslant T]] + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 r) \lim_{T\to\infty}(\sup[p(t): t \geqslant T])$$

$$+ A \lim_{T\to\infty}\{\sup_{x\in U}[\sup(\|x_t\|_{L^2} p(t): t \geqslant T)]\}$$

$$+\lim_{T\to\infty}\sup\Big[p(t)\int_0^t |||k(t,\tau)|||\, v(\tau)\,d\tau:\ t \geqslant T\Big]$$

$$\leqslant (\varepsilon_1+\varepsilon_2+\varepsilon_3 r)C + A\lim_{T\to\infty}\Big\{\sup_{x\in U}\big[\sup(\|x_t\|_{L^2}\, p(t):\ t \geqslant T)\big]\Big\}.$$

Letting now $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, $\varepsilon_3 \to 0$, we get (3.14). Finally, by virtue of (3.13) and (3.14) we obtain

$$\mu_0(HU) \leqslant A\mu_0(U)$$

which proves that H is a μ_0-contraction.

In order to complete our proof it suffices to apply Theorem 2.1.

COROLLARY 3.1. *Let the stochastic functional-integral equation* (1.1) *satisfy* (i), (ii), (v) *and* (vi) *of Theorem* 3.1. *Moreover, suppose that*

$$\text{(K)} \qquad |||k(t,\tau)||| \leqslant k_1(t)\,k_2(\tau),$$

where k_1 *is a positive differentiable function,* k_2 *is a positive function,*

$$\text{(iii}_1) \qquad k_1(t)\int_0^t k_2(\tau)\,u(\tau)\,d\tau \leqslant A, \qquad t\in \mathbf{R}^+,\ 0\leqslant A<1,$$

and

$$\text{(iv}_1) \qquad \int_0^t k_2(\tau)\,p(\tau)\,v(\tau)\,d\tau = B, \qquad t\in \mathbf{R}^+,\ 0\leqslant B<\infty.$$

If p *is nonincreasing then*

$$p(t) = \big(-Bk_1'(t)\big)/\big(k_1^2(t)\,k_2(t)\,v(t)\big) \tag{3.15}$$

and

$$\|x(t)\|_{L^2} = o\Big(\big(k_1^2(t)\,k_2(t)\,v(t)\big)/k_1'(t)\Big), \qquad \text{as } t\to\infty. \tag{3.16}$$

Proof. If p is a nonincreasing function and (K), (iii$_1$), (iv$_1$) are satisfied, then (iii) and (iv) hold too. From (iv$_1$) we get (3.15), while (3.16) is a consequence of Theorem 3.1.

Remark. Note that under the assumptions of Corollary 3.1 with $v(t) \equiv 0$, we have

$$\|x(t)\|_{L^2} = o\big(1/p(t)\big), \qquad t\to\infty,$$

for any nonincreasing function p such that (v) and (vi) are satisfied.

COROLLARY 3.2. *Suppose that the assumptions* (i), (ii), (v), (vi) *of Theorem* 3.1 *and the assumption* (K) *of Corollary* (3.1) *is satisfied. Let*

$$\text{(iii}_2) \qquad w(t) = p(t)\,k_1(t)\int_0^t k_2(\tau)\big(u(\tau)/p(\tau)\big)\,d\tau$$

be positive, differentiable,

$$\sup\{w(t)\colon t\in \boldsymbol{R}^+\} = D, \quad 0 \leqslant D < 1,$$

and

$$w(t)v(t) \leqslant k_1(t)u(t), \quad t\in \boldsymbol{R}^+.$$

Then

$$p(t) = [w(t)/k_1(t)]\exp\Big(-\int_0^t \big(k_1(\tau)k_2(\tau)u(\tau)/w(\tau)\big)d\tau\Big) \tag{3.17}$$

and

$$\|x(t)\|_{L^2} = o\Big([k_1(t)/w(t)]\exp\Big(\int_0^t \big(k_1(\tau)k_2(\tau)u(\tau)/w(\tau)\big)d\tau\Big)\Big) \quad \textit{as } t\to\infty. \tag{3.18}$$

Proof. (3.17) follows from (iii_2) and (3.18) is a consequence of Theorem 3.1. Moreover, we see that (iii_2) implies (iii) and (iv_2) gives (iv) as by (iv_2)

$$\begin{aligned} p(t)\int_0^t |||k(t,\tau)|||\,v(\tau)d\tau &\leqslant p(t)k_1(t)\int_0^t k_2(\tau)v(\tau)d\tau \\ &\leqslant \exp\Big(\int_0^t k_2(\tau)v(\tau)d\tau - \int_0^t \big(k_1(\tau)k_2(\tau)u(\tau)/w(\tau)\big)d\tau\Big) \\ &= \exp\Big(\int_0^t \big(k_2(\tau)\big(v(\tau)w(\tau)-k_1(\tau)u(\tau)\big)/w(\tau)\big)d\tau\Big) \leqslant 1, \end{aligned}$$

which completes the proof.

We now consider the stochastic functional-integral equation (1.2).

THEOREM 3.2. *Let the stochastic functional-integral equation* (1.2) *satisfy the following conditions*:

(i) $$|f(t, x_t(\omega))| \leqslant u_1(t)|x_t(\omega)| + v_1(t) \quad \mathscr{P}\text{-a.s.}$$

where the functions u_1 and v_1 defined for $t\in\boldsymbol{R}^+$ are nonnegative and continuous,

(ii) *as in Theorem* 3.1,

(iii) $p(t)\int_0^\infty |||k(t-\tau)|||\big(u_1(\tau)/p(\tau)\big)d\tau \leqslant A_1, \quad t\in\boldsymbol{R}^+$, *where* $A_1\in[0, 1)$,

(iv) $p(t)\int_0^\infty |||k(t-\tau)|||\,v_1(\tau)d\tau \leqslant B_1, \quad t\in\boldsymbol{R}^+$, *and* $0\leqslant B_1 < +\infty$,

(v) $\lim_{t\to\infty} p(t) \int_0^\infty |||k(t-\tau)|||\, v_1(\tau)\, d\tau = 0, \quad \lim_{t\to\infty} p(t)\, \|h(t)\|_{L^2} = 0,$

(vi) *as in Theorem* 3.1,

(vii) $m\left(\int_0^\infty k(t-\tau;\omega) f(\tau, U(\tau))\, d\tau\right) = 0,$ *for any* $t \geq 0$ *and* $U \in \mathfrak{M}_{C_p}$.

Then there exists at least one solution $x \in C_p$ *of* (1.2) *such that*

$$\lim_{t\to\infty} p(t)\, \|x(t)\|_{L^2} = 0.$$

Proof. Define the map F on C_p by

$$(Fx)(t;\omega) = h(t;\omega) + \int_0^\infty k(t-\tau;\omega) f(\tau, x_\tau(\omega))\, d\tau.$$

Analogously as in Theorem 3.1 we get

$$\|Fx\|_p \leq \|h\|_p + A_1\, \|x_p\| + B_1$$

which implies that F maps C_p in C_p. Moreover, we note that

$$F\colon K(\Theta, r) \to K(\Theta, r) \quad \text{for } r = (\|h\|_p + B_1)/(1-A_1).$$

We now prove that F is continuous in $K(\Theta, r)$. Let $x, y \in K(\Theta, r)$, without loss of generality we may assume that $u_1(t) \geq 1$, $t \in \boldsymbol{R}^+$. Let $T > 0$ be fixed and $t \in [0, T]$. Taking into account that f is uniformly continuous on $[0, T] \times K(\Theta, r)$, (vi), we have for any given $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$,

$$\begin{aligned}
\|Fx - Fy\|_p &\leq p(t) \int_0^\infty |||k(t-\tau)|||\, \|f(\tau, x_\tau) - f(\tau, y_\tau)\|_{L^2}\, d\tau \\
&= p(t) \int_0^T |||k(t-\tau)|||\, \|f(\tau, x_\tau) - f(\tau, y_\tau)\|_{L^2}\, d\tau \\
&\quad + p(t) \int_T^\infty |||k(t-\tau)|||\, \|f(\tau, x_\tau) - f(\tau, y_\tau)\|_{L^2}\, d\tau \\
&\leq \varepsilon_1 A_1 + p(t) \int_T^\infty |||k(t-\tau)|||\, \big(u_1(\tau)/p(\tau)\big)\, \|f(\tau, x_\tau) - f(\tau, y_\tau)\|_{L^2}\, p(\tau)\, d\tau \\
&\leq \varepsilon_1 A_1 + \varepsilon_2 A_1
\end{aligned}$$

which proves that F is continuous. Now put

$$\begin{aligned}
g_t(\tau) &= |||k(t-\tau)|||\, \big(u_1(\tau)/p(\tau)\big), \\
g_t^*(\tau) &= g(\tau, t)\, \|x_\tau\|_{L^2}\, p(\tau), \quad t \in \boldsymbol{R}^+,\ x \in K(\Theta, r).
\end{aligned}$$

By the assumptions $\int_0^\infty g_t(\tau)\,d\tau < \infty$, $\int_0^\infty g_t^*(\tau)\,d\tau < \infty$, $t \in \boldsymbol{R}^+$. This fact allows us to find functions $\tilde{g}_t$, $\tilde{g}_t^*$ which are nonnegative, decreasing and vanishing at infinity such that

$$g_t(\tau) \leqslant \tilde{g}_t(\tau), \quad g_t^*(\tau) \leqslant \tilde{g}_t^*(\tau),$$

$$\int_0^\infty \tilde{g}_t(\tau)\,d\tau < \infty, \quad \int_0^\infty \tilde{g}_t^*(\tau)\,d\tau < \infty.$$

Hence, we can write

$$\int_0^\infty \tilde{g}_t(\tau)\,d\tau = \lim_{h\to 0^+} h \sum_{n=1}^\infty \tilde{g}_t(nh)$$

and

$$\int_0^\infty \tilde{g}_t^*(\tau)\,d\tau = \lim_{h\to 0^+} h \sum_{n=1}^\infty \tilde{g}_t^*(nh).$$

Moreover, $\tilde{g}_t$ can be chosen such that

$$\lim_{0<h\to 0} h \left| \sum_{n=1}^\infty \tilde{g}_t(nh) - \sum_{n=1}^\infty g_t(nh) \right| = 0$$

and

$$\lim_{0<h\to 0} h \left| \sum_{n=1}^\infty \tilde{g}_t^*(nh) - \sum_{n=1}^\infty g_t^*(nh) \right| = 0.$$

Now fix $U \subset K(\Theta, r)$. We prove that

$$a(FU) \leqslant A_1\, a(U). \tag{3.19}$$

Let $T > 0$ be fixed. Choose m large enough so that $m+1 > T$. Then by the assumptions we have

$$\begin{aligned}
\|(Fx)(t)\|_{L^2}\, p(t) &\leqslant \|h(t)\|_{L^2}\, p(t) + p(t) \int_0^\infty g_t^*(\tau)\,d\tau \\
&\quad + p(t) \int_0^\infty |||k(t-\tau)|||\, v_1(\tau)\,d\tau \\
&\leqslant \|h(t)\|_{L^2}\, p(t) + p(t) \left| \int_0^\infty \tilde{g}_t^*(\tau)\,d\tau - h \sum_{n=1}^\infty \tilde{g}_t^*(nh) \right|
\end{aligned}$$

$$+p(t)\,h\left|\sum_{n=1}^{\infty}\tilde{g}_t^*(nh)-\sum_{n=1}^{\infty}g_t^*(nh)\right|+p(t)\,h\sum_{n=1}^{\infty}g_t^*(nh)$$

$$+p(t)\int_0^{\infty}|||k(t-\tau)|||\,v_1(\tau)\,d\tau$$

$$\leqslant \|h(t)\|_{L^2}\,p(t)+p(t)\left|\int_0^{\infty}\tilde{g}_t^*(\tau)\,d\tau-h\sum_{n=1}^{\infty}\tilde{g}_t^*(nh)\right|$$

$$+p(t)\,h\left|\sum_{n=1}^{\infty}\tilde{g}_t^*(nh)-\sum_{n=1}^{\infty}g_t^*(nh)\right|+p(t)\,hr\sum_{n=1}^{m}g_t(nh)$$

$$+p(t)\sup\left[\|x_{nh}\|_{L^2}\,p(nh):\ n\geqslant m+1\right]h\sum_{n=m+1}^{\infty}g_t(nh)$$

$$+p(t)\int_0^{\infty}|||k(t-\tau)|||\,v_1(\tau)\,d\tau$$

$$\leqslant \|h(t)\|_{L^2}\,p(t)+p(t)\left|\int_0^{\infty}\tilde{g}_t(\tau)\,d\tau-h\sum_{n=1}^{\infty}\tilde{g}_t^*(nh)\right|$$

$$+p(t)\,h\left|\sum_{n=1}^{\infty}\tilde{g}_t^*(nh)-\sum_{n=1}^{\infty}g_t^*(nh)\right|+hrp(t)\sum_{n=1}^{m}g_t(nh)$$

$$+p(t)\sup\left[\|x_t\|_{L^2}\,p(t):\ t\geqslant T\right]h\sum_{n=1}^{\infty}g_t(nh)$$

$$+p(t)\int_0^{\infty}|||k(t-\tau)|||\,v_1(\tau)\,d\tau.$$

Letting now $h\to 0$, we get

$$\|(Fx)(t)\|_{L^2}\,p(t)\leqslant \|h(t)\|_{L^2}\,p(t)$$

$$+\sup\left[\|x_t\|_{L^2}\,p(t):\ t\geqslant T\right]p(t)\int_0^{\infty}g_t(\tau)\,d\tau+p(t)\int_0^{\infty}|||k(t-\tau)|||\,v_1(\tau)\,d\tau.$$

Hence, by the definition of a, we obtain (3.19).

Now let be given $\varepsilon>0$, $T>0$, and $t, s\in[0, T]$, $|t-s|\leqslant\varepsilon$. By (i), (iii), for $x\in U\subset K(\Theta, r)$ we have

$$\|(Fx)(t)\,p(t)-(Fx)(s)\,p(s)\|_{L^2}\leqslant |p(t)-p(s)|\,\|h(t)\|_{L^2}+p(s)\,\|h(t)-h(s)\|_{L^2}$$

$$+|p(t)-p(s)|\int_0^{\infty}|||k(t-\tau)|||\,\|f(\tau, x_\tau)\|_{L^2}\,d\tau$$

$$+p(s)\int_0^{\infty}|||k(t-\tau)-k(s-\tau)|||\ \|f(\tau, x_\tau)\|_{L^2}\,d\tau.$$

Notice that by the assumptions (iii) and (iv)

$$|p(t)-p(s)|\int_0^\infty |||k(t-\tau)|||\,\|f(\tau, x_\tau)\|_{L^2}\,d\tau$$

$$\leqslant |p(t)-p(s)|\Big(r\int_0^\infty |||k(t-\tau)|||\big(u_1(\tau)/p(\tau)\big)\,d\tau+\int_0^\infty |||k(t-\tau)|||\,v_1(\tau)\,d\tau\Big)$$

$$\leqslant \min\{p(t)^{-1}\colon\ 0\leqslant t\leqslant T\}\,|p(t)-p(s)|\,(rA_1+B_1).$$

Now

$$p(s)\int_0^\infty |||k(t-\tau)-k(s-\tau)|||\,\|f(\tau, x_\tau)\|_{L^2}\,d\tau$$

$$\leqslant 2\{\min p(t)^{-1}\colon\ 0\leqslant t\leqslant T\}\,p(s)(A_1 r+B_1).$$

Using the above, for any given $\varepsilon_1>0$, $\varepsilon_2>0$ and sufficiently large T_1, we have

$$p(s)\int_0^\infty |||k(t-\tau)-k(s-\tau)|||\,\|f(\tau, x_\tau)\|_{L^2}\,d\tau$$

$$\leqslant rp(s)\int_0^{T_1} |||k(t-\tau)-k(s-\tau)|||\big(u_1(\tau)/p(\tau)\big)\,d\tau$$

$$+p(s)\int_0^{T_1} |||k(t-\tau)-k(s-\tau)|||\,v_1(\tau)\,d\tau$$

$$+p(s)\int_{T_1}^\infty |||k(t-\tau)-k(s-\tau)|||\,\|f(\tau, x_\tau)\|_{L^2}\,d\tau$$

$$\leqslant \varepsilon_1 rp(s)\,T_1\{\max u_1(\tau)/p(\tau)\colon\ 0\leqslant\tau\leqslant T_1\}$$

$$+\varepsilon_1 p(s)\,T_1\max\{v_1(\tau)\colon\ 0\leqslant\tau\leqslant T_1\}+\varepsilon_2 p(s).$$

Hence, we conclude that

$$\beta_0(FU)=0. \tag{3.20}$$

Therefore, by (3.19) and (3.20), we obtain

$$\mu_0(FU)\leqslant A_1\,\mu_0(U)$$

which proves that F is μ_0-contraction and completes the proof.

Remarks. In [8] the following theorem is proved.

THEOREM A. *Let the stochastic functional-integral equation*

$$x(t;\omega)=h(t;\omega)+\int_0^t k(t,\tau;\omega)\,f\big(\tau, x_\tau(\omega)\big)\,d\tau \tag{3.21}$$

satisfy the following conditions:

(i) *the mapping* $x(t;\omega) \to f(t, x_t(\omega))$ *is a completely continuous map from* $C_c(\boldsymbol{R}^+, L^2(\Omega, \mathfrak{A}, \mathscr{P}))$ *into* $C_c(\boldsymbol{R}^+, L^2(\Omega, \mathfrak{A}, \mathscr{P}))$,

(ii) *there exist two continuous non-negative real functions* $g(t)$ *and* $l(t)$ *defined for* $t \in \boldsymbol{R}^+$ *such that*

(a) $$\|f(t, x_t(\omega))\|_{L^2} \leqslant l(t), \quad \text{whenever } \|x(t;\omega)\|_{L^2} \leqslant g(t)$$

and

(b) $$\|h(t;\omega)\|_{L^2} + \int_0^t |||k(t,\tau)|||\, l(\tau)\,d\tau \leqslant g(t), \qquad t \in \boldsymbol{R}^+.$$

Then there exists at least one solution $x(t;\omega)$ *of* (3.21) *in the space* $C_c(\boldsymbol{R}^+, L^2(\Omega, \mathfrak{A}, \mathscr{P}))$ *such that*

$$\|x(t;\omega)\|_{L^2} \leqslant g(t).$$

We now show that the assumptions of Theorem A imply those ones of Theorem 3.1.

From the assumptions of Theorem A we conclude that without loss of generality we can choose a function $g(t)$ such that $\lim_{t\to\infty} g(t) = \infty$ and $\sup_{t\geqslant 0} \|h(t)\|_{L^2}/g(t) = 1 - A$ (A will be defined later), $\lim_{t\to\infty} (1/g(t))\|h(t)\|_{L^2} = 0$.

Now put $p(t) = 1/g(t)$, $u(t) = l(t)/g(t)$, $v(t) \equiv 0$. It is easy to see that (i) of Theorem A implies (i) and (vi) of Theorem 3.1. (iii) and (iv) are trivially satisfied. The condition (vi) is satisfied by the above choice of g. From the assumption (b) we conclude in turn

$$(l(t)/g(t))\|h(t)\|_{L^2} + (l(t)/g(t)) \int_0^t |||k(t,\tau)|||\, u(\tau)/p(\tau)\,d\tau \leqslant 1,$$

$$\sup_{t\geqslant 0} p(t)\|h(t)\|_{L^2} + \sup_{t\geqslant 0} (1/g(t)) \int_0^t |||k(t,\tau)|||\, u(\tau)/p(\tau)\,d\tau \leqslant 1,$$

$$1 - A + \sup_{t\geqslant 0} p(t) \int_0^t |||k(t,\tau)|||\, u(\tau)/p(\tau)\,d\tau \leqslant 1,$$

$$\sup_{t\geqslant 0} p(t) \int_0^t |||k(t,\tau)|||\, u(\tau)/p(\tau)\,d\tau \leqslant A,$$

which proves (ii) of Theorem 3.1.

We now consider the stochastic functional-integral equation (1.2) of [8].

THEOREM B. *Let the stochastic functional-equation* (1.2) *satisfy the following conditions:*

(i) $$\sup_{t\geqslant 0} \|h(t,\omega)\|_{L^2} < M, \qquad M \in \boldsymbol{R}^+,$$

(ii) $k(t,\omega)$ *is* $\mathscr{P}$-*essentially bounded and continuous as a map from* $\boldsymbol{R}^+$

into $L_\gamma(\Omega, \mathfrak{A}, \mathscr{P})$ *such that*

$$K = \int_0^\infty |||k(t, \omega)||| \, dt < \infty,$$

(iii) f: $x(t;\omega) \to f(t, x_t(\omega))$ *is a completely continuous map from* $C_c(\boldsymbol{R}^+, L^2(\Omega, \mathfrak{A}, \mathscr{P}))$ *into* $C_c(\boldsymbol{R}^+, L^2(\Omega, \mathfrak{A}, \mathscr{P}))$ *such that* $\|x(t;\omega)\|_{L^2} \leqslant M + K\varphi(M)$ *imply* $\|f(t, x_t(\omega))\|_{L^2} \leqslant \varphi(M)$, $t \in \boldsymbol{R}^+$, *and* $\varphi(M)$ *is a nonnegative real valued function defined for sufficiently large* M.

Then there exists at least one solution $x(t;\omega)$ *of* (1.2) *in the space* $C_c(\boldsymbol{R}^+, L^2(\Omega, \mathfrak{A}, \mathscr{P}))$ *such that* $\|x(t, \omega)\|_{L^2}$ *is bounded.*

We now show that the conditions of Theorem B imply the conditions of Theorem 3.2. Indeed, putting $p(t) = 1/(M + K\varphi(M))$, $u_1(t) \equiv 0$, $v_1(t) = \varphi(M)$, we easily see that the hypotheses (i)–(vii) in Theorem 3.1 are satisfied.

REFERENCES

[1] J. Banaś, *Measure of noncompactness in the space of continuous tempered functions*, Demonstratio Mathematica 14 (1981), p. 127–133.

[2] – and K. Goebel, *Measures of noncompactness in Banach space*, Lecture Notes in Pure and Applied Mathematics 60, Marcel Dekker, New York and Basel 1980.

[3] A. T. Bharucha-Reid, *Random Integral Equations*, New York 1972.

[4] –, *Lectures on Theory of Random Equations*, Madras 1963.

[5] W. J. Padgett, *On the existence of a solution of integral equation in turbulence theory*, Journal of Mathematical Physics 12 (1971), p. 210–212.

[6] –, *On a stochastic integral equation of the Volterra type in telephone traffic theory*, Journal of Applied Probability 8 (1971), p. 269–275.

[7] – and C. P. Tsokos, *On a semi-stochastic model arising in a biological system*, Mathematical Biosciences 9 (1970), p. 105–117.

[8] A. N. V. Rao and C. P. Tsokos, *On a class of stochastic functional integral equations*, Colloquium Mathematicum 35 (1976), p. 141–146.

[9] C. P. Tsokos, *The method of V. M. Popov for differential systems with random parameters*, Journal of Applied Probability 8 (1971), p. 298–310.

[10] –, *On some nonlinear differential systems with random parameters*, in: *IEEE Proceedings of the Third Annual Princeton Conference on Informatical Sciences and Systems, 1969*, p. 228–234.

[11] K. Zima, *Sur l'existence des solutions d'une équation intégro-différentialle*, Annales Polonici Mathematici 27 (1973), p. 181–187.

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