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# GRAPHS WITH PRESCRIBED SIZE AND VERTEX PARITIES

BY

PHILIP J. PRATT AND DONALD W. VANDERJAGT
(ALLENDALE, MICHIGAN)

The *degree sequence* of a graph is defined as the sequence formed by the non-negative integers which are the degrees of the vertices of the graph. The question as to whether or not a graph exists with a given sequence as its degree sequence has been investigated; a constructive answer has been given independently by Havel [2] and Hakimi [1]. More recently this question has been generalized by investigating (see [3] and [4]) whether or not there is a graph with a given *degree set* (the set of integers in the degree sequence). In this paper we further generalize by investigating whether or not there is a graph with a prescribed number of even and odd vertices. With no further conditions imposed, the answer is trivial since the graph $\bar{K}_m \cup (n/2)K_2$ has $m$ even and $n$ odd vertices. Thus we impose restrictions on the size of the graph. For non-negative integers $m$ and $n$, with $n$ even, an *$m$: $n$ graph* is defined to be a graph with $m$ even vertices and $n$ odd vertices. The problem we investigate is: for which integers $k$ is there an $m$: $n$ graph of size $k$, that is, with $k$ edges?

THEOREM. *Let $m$ and $n$ be non-negative integers with $n$ even. There is an $m$: $n$ graph of size $k$ if and only if*

$$\frac{n}{2} \leqslant k \leqslant \binom{m+n}{2} - d, \quad \text{where } d = \begin{cases} m/2 & \text{if } m \text{ is even,} \\ n/2 & \text{if } m \text{ is odd,} \end{cases}$$

*with the following exceptions:*

*if $n = 0$, there is no such graph of size 1 or 2;*

*if $n = 0$ and $m$ is odd, there is no such graph of size*

$$\binom{m}{2} - 2 \quad \text{and} \quad \binom{m}{2} - 1.$$

Proof. The proof will be divided into four cases, determined by the values of $m$ and $n$.

Case 1. Let $m, n \geqslant 2$.

The graph $\overline{K}_m \cup (n/2)K_2$ has the minimum size, namely $n/2$. The maximum size,

$$\binom{m+n}{2} - d,$$

is obtained by deleting $d$ mutually non-adjacent edges from $K_{m+n}$. We now show that for each integer $k$ between

$$\frac{n}{2} \quad \text{and} \quad \binom{m+n}{2} - d$$

there is an $m: n$ graph of size $k$. Let $k_0$ be the smallest integer for which there is an $m: n$ graph of size $k_0$ but no such graph of size $k_0+1$. Then $k_0 \geqslant n/2$. Let $G$ be an $m: n$ graph of size $k_0$, with $A$ the set of all even vertices of $G$, and $B$ the set of all odd vertices of $G$. We complete the proof in this case by showing that

$$k_0 = \binom{m+n}{2} - d.$$

Since there is no $m: n$ graph of size $k_0+1$, each vertex in $A$ is adjacent to each vertex in $B$. Furthermore, all the vertices in $A$ are adjacent or all the vertices in $B$ are adjacent, for otherwise, if $u, v \in A$ are non-adjacent and $w, x \in B$ are non-adjacent, then $G - uw + uv + wx$ is an $m:n$ graph of size $k_0+1$. (In the resulting graph, $v \in B$ and $x \in A$.) Thus, exactly one of $A$ and $B$ has the property that the subgraph induced by it is the complete graph; suppose that $A$ has this property. In $B$, no vertex has degree less than $(m+n-1)-1$, for suppose $v \in V(G)$ is such that $\deg v \leqslant m+n-3$. Then there are vertices $u$ and $w$ in $B$ for which $uv, wv \notin E(G)$. If $uw \in E(G)$, then $G - uw + uv + wv$ is an $m: n$ graph of size $k_0+1$. If $uw \notin E(G)$, let $r, s \in A$. Then $G - ur - ws + uv + uw + wv$ is an $m: n$ graph of size $k_0+1$. Thus $G$ is the graph $K_{m+n}$ with some non-adjacent edges removed. Since $G$ is an $m: n$ graph, there must be $d$ edges removed, so that $G$ has size

$$\binom{m+n}{2} - d.$$

Thus

$$k_0 = \binom{m+n}{2} - d.$$

Case 2. Let $n = 0$.

In this case the graph $G$ has all even vertices. Thus $\overline{K}_m$ has the minimum size. Clearly, there is no $m: 0$ graph of size 1 or 2. Also, if

$m$ is odd, there is a graph of size $\binom{m}{2}$, namely $K_m$, but none of size

$$\binom{m}{2}-1 \quad \text{or} \quad \binom{m}{2}-2.$$

The graph $\bar{K}_{m-3} \cup K_3$ has the size 3. Let $k_0$ be the smallest positive integer for which there is an $m\colon 0$ graph of size $k_0$ but no such graph of size $k_0+1$. Then $k_0 \geqslant 3$. Let $G$ be an $m\colon 0$ graph of size $k_0$. We complete the proof is this case by showing that

$$k_0 = \binom{m}{2} - c, \quad \text{where } c = \begin{cases} m/2 & \text{if } m \text{ is even,} \\ 3 & \text{if } m \text{ is odd.} \end{cases}$$

First note that each vertex in $G$ must be adjacent to at least one incident vertex of each edge of $G$, for if $u, v, w \in V(G)$ and $uw \in E(G)$ but $uv, vw \notin E(G)$, then $G - uw + uv + vw$ is an $m\colon 0$ graph of size $k_0+1$. Thus, for any three vertices of $G$, if two pairs of these vertices are non-adjacent, so is the third pair. Furthermore, since $k_0 \geqslant 3$, no vertex is isolated. If $G$ contains a triple of mutually non-adjacent vertices, then all other vertices of $G$ are adjacent to each vertex in this triple; suppose not. Then there are vertices $v, u, w, x$ of $G$ for which $uw, ux, wx \notin E(G)$ and $v$ is not adjacent either to $u$, $w$ or to $x$. But $v$ is not isolated, so there is a vertex $t$ of $G$ for which $vt \in E(G)$. Thus $ut, wt, xt \in E(G)$, so $G - ut - wt + vw + vx + ux$ is an $m\colon 0$ graph of size $k_0+1$. Thus $v$ is adjacent to at least one of $u$, $w$, and $x$, so, as above, $v$ is adjacent to each of $u$, $w$, and $x$, as claimed. Finally, if $G$ contains three mutually non-adjacent vertices, then each other pair of vertices in $G$ must be adjacent. If not, let $u$, $v$, and $w$ be mutually non-adjacent vertices in $G$ and let $s, t \in V(G)$. Since each of $u$, $v$, $w$ is adjacent to $s$ and $t$, if $st \notin E(G)$, then $G - su - tv + st + uw + vw$ is an $m\colon 0$ graph of size $k_0+1$. Thus $st \in E(G)$.

In summary, there are three possibilities for $G$, namely, $K_m$, $K_m - K_3$, or $K_m$, with some non-adjacent edges removed. If $m$ is even, the $m\colon 0$ graph $G$ must be $K_m$ with exactly $m/2$ edges removed; that is,

$$k_0 = \binom{m}{2} - \frac{m}{2}.$$

If $m$ is odd, then $G = K_m$ or $G = K_m - K_3$. By the definition of $k_0$, we have

$$k_0 = \binom{m}{2} - 3.$$

In either case,

$$k_0 = \binom{m}{2} - c.$$

Case 3. Let $m = 0$.

Let $G$ be a $0\colon n$ graph of size $k$, where $n$ is an even integer. Then $\bar{G}$ is an $n\colon 0$ graph of size

$$\binom{n}{2} - k.$$

By Case 2, this is possible if and only if

$$k = \frac{n}{2}, \frac{n}{2}+1, \ldots, \binom{n}{2}-4, \binom{n}{2}-3, \binom{n}{2}.$$

Case 4. Let $m = 1$.

A $1\colon n$ graph has $n+1$ vertices exactly one of which has even degree, where $n$ is an even integer. Thus $K_1 \cup (n/2)K_2$ is a $1\colon n$ graph of smallest size, namely $n/2$. The maximum size,

$$\binom{n}{2} - \frac{n}{2},$$

is obtained by deleting $n/2$ mutually non-adjacent edges from $K_{1+n}$. For $n = 2$, a graph may have the size 1 or 2. Let $n \geqslant 4$ and let $k$ be an integer, where $n/2 \leqslant k \leqslant n^2/2$. Since $n \geqslant 4$,

$$\frac{n+2}{2} \leqslant \binom{n}{2} - 3 \quad \text{and} \quad \binom{n}{2} + 1 \geqslant n+3,$$

so that $k$ has to satisfy at least one of the following conditions:

$$\frac{n}{2} \leqslant k \leqslant \binom{n}{2} - 3, \tag{1}$$

$$\frac{n+2}{2} \leqslant k \leqslant \binom{n}{2} + 1, \tag{2}$$

$$n+3 \leqslant k \leqslant \frac{n^2}{2} = \binom{n}{2} + \frac{n}{2}. \tag{3}$$

If $k$ satisfies (1), there is, by Case 3, a $0\colon n$ graph $G'$ of size $k$. Addition of a vertex of degree 0 to $G'$ produces a $1\colon n$ graph $G$ of size $k$. If $k$ satisfies (2), then

$$\frac{n-2}{2} \leqslant k-2 \leqslant \binom{n}{2} - 1,$$

so, by Case 1, there is a $2\colon n-2$ graph $G'$ of size $k-2$. Let $u, v \in V(G')$ be the even vertices and let $w \notin V(G')$. Let $G$ be the graph for which

$$V(G) = V(G') \cup \{w\} \quad \text{and} \quad E(G) = E(G') \cup \{uw, vw\}.$$

The graph $G$ is a $1:n$ graph of size $k$. If $k$ satisfies (3), then

$$3 \leqslant k-n \leqslant \binom{n}{2} - \frac{n}{2},$$

so there is an $n:0$ graph $G'$ of size $k-n$. Then $G = G'+K_1$ is a $1:n$ graph of size $k$.

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