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CAUCHY'S FUNCTIONAL EQUATION ON A RESTRICTED DOMAIN

BY

MAREK KUCZMA (KATOWICE)

In [2] Baron and Ger have reduced the Mikusiński-Pexider equation to the Cauchy equation on restricted domain,

$$f(x+y) = f(x)+f(y) \quad \text{for } x \in G,\, y \in Y, \tag{1}$$

for functions $f\colon G \to H$. Here G and H are arbitrary groups (written additively), whereas Y is a non-empty subset of G.

The authors of [2] have proved the following result regarding equation (1) in the case where the groups G, H are abelian:

LEMMA 1. *If, for every $q \in G$,*

$$Y \cap (Y-q) \neq \emptyset, \tag{2}$$

then the function f satisfies $f(x+y) = f(x)+f(y)$ for all $x, y \in G$.

The purpose of the present note is to solve equation (1) under less restrictive conditions. In particular, we do not assume that the groups G and H are abelian (as it was the case in [2]). However, because of the connection of the problem treated with those dealt with in [2], we preserve the additive notation.

Write

$$Y^* = \{y \in G\colon \bigvee_{x \in G} f(x+y) = f(x)+f(y)\}.$$

Evidently,

$$Y \subset Y^*. \tag{3}$$

The following lemma is crucial for further considerations:

LEMMA 2. *Y^* is a subgroup of G.*

Proof. Take in (1) arbitrary $y \in Y$ and $x = 0$. We get $f(y) = f(0) + f(y)$, whence $f(0) = 0$. Thus we get for arbitrary $y \in Y^*$

$$f(-y)+f(y) = f(0) = 0,$$

i.e.,

$$f(-y) = -f(y) \quad \text{for } y \in Y^*. \tag{4}$$

Then we have for $x \in G$, $y \in Y^*$

$$f(x) = f[(x-y)+y] = f(x-y)+f(y),$$

whence $f(x-y) = f(x)-f(y)$, or, by (4),

$$f[x+(-y)] = f(x)+f(-y) \quad \text{for } x \in G,\ y \in Y^*.$$

This means that for $y \in Y^*$ also $-y \in Y^*$.

Now take arbitrary $u, v \in Y^*$. Evidently,

$$f(u+v) = f(u)+f(v). \tag{5}$$

Thus we have for arbitrary $x \in G$

$$f(x+u+v) = f(x+u)+f(v) = f(x)+f(u)+f(v)$$

and, by (5),

$$f(x+u+v) = f(x)+f(u+v).$$

This means that $u+v \in Y^*$ and completes the proof of the lemma.

Now we may give a simple proof of Lemma 1 (without assuming the commutativity of the group operations in G and H). Conditions (2) and (3) imply that, for every $q \in G$, $Y^* \cap (Y^*-q) \neq \emptyset$. This means that every $q \in G$ may be written as $q = u-v$ with $u, v \in Y^*$, i.e., $q \in Y^*$. Hence $Y^* = G$.

In view of Lemma 2 we may assume in the sequel that Y is a subgroup of G.

The following theorem yields the main result of the present paper:

THEOREM. *Let G and H be (not necessarily commutative) groups and let $Y \neq \emptyset$ be a subgroup of G. If every homomorphism g_0: $Y \to H$ can be extended to a homomorphism $\bar{g}_0$: $G \to H$, then the general solution of equation* (1) *can be written in the form*

$$f(x) = h(x)+g(x), \quad x \in G, \tag{6}$$

where g: $G \to H$ is an arbitrary homomorphism such that

$$g(y) = f(y) \quad \text{for } y \in Y, \tag{7}$$

and h is an arbitrary function which is constant on the left cosets of G with respect to Y and $h(y) = 0$ for $y \in Y$.

Proof. Suppose that a function f: $G \to H$ satisfies equation (1). Thus f restricted to Y is a homomorphism, which by hypothesis admits an extension to a homomorphism g: $G \to H$,

$$g(x+y) = g(x)+g(y) \quad \text{for } x, y \in G. \tag{8}$$

Put

$$h(x) = f(x)-g(x) \quad \text{for } x \in G. \tag{9}$$

Then $h(y) = 0$ for $y \in Y$. Moreover, if u and v are elements of the same left coset of G with respect to Y, then $v = u+y$ with a $y \in Y$ and

$$\begin{aligned} h(v) &= h(u+y) = f(u+y)-g(u+y) = [f(u)+f(y)]-[g(u)+g(y)] \\ &= f(u)+f(y)-g(y)-g(u) = f(u)-g(u) = h(u) \end{aligned}$$

by (9), (1), (8) and (7). Thus h has the announced properties and (6) results from (9).

Conversely, suppose that f has form (6) with suitable h and g. Take arbitrary $x \in G$, $y \in Y$. Then $h(x+y) = h(x)$ and, by (8) and (6), we obtain

$$f(x+y) = h(x+y)+g(x+y) = h(x)+g(x)+g(y) = f(x)+f(y),$$

since $h(y) = 0$. This completes the proof.

The assumption regarding the existence of an extension g of g_0 is fulfilled, in particular, in the following cases.

(i) *Y has a finite index n and the division by n is performable in H.*

In fact, $g(x) = g_0(nx)/n$ is the desired extension. (It is unique in this case.)

(ii) *G and H are abelian and H is divisible* (cf., e.g., [1], Lemma 1.5).

Case (ii) contains, in particular, the case where G is abelian and H is the additive group of real or complex numbers.

The following example shows that the extensibility hypothesis in our theorem (as well as the divisibility hypothesis in case (i) above) is essential.

Let $G = H = Z$ be the additive group of integers and let $Y = 2Z$ be the additive group of even integers. The function f defined by

$$f(2k) = k, \quad f(2k+1) = k+1, \quad k = 0, \pm 1, \pm 2, \ldots,$$

satisfies equation (1), but cannot be written in form (6) with integer-valued functions h, g. As it is easily seen, the homomorphism $g_0: 2Z \to Z$ defined by $g_0(2k) = k$ does not admit an extension to a homomorphism $g: Z \to Z$.

REFERENCES

[1] S. Balcerzyk, *Wstęp do algebry homologicznej*, Warszawa 1970.

[2] K. Baron and R. Ger, *On Mikusiński-Pexider functional equation*, this fascicle, p. 307-312.

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