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ON QUALITATIVE CLUSTER SETS

BY

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1. Introduction. Let f be a function from the open upper half plane H into the Riemann sphere W. A well-known result of Collingwood (see [4], Theorem 3, p. 8) states that for a continuous function f nearly every x in the real line R has the property that for nearly every direction $\theta \in (0, \pi)$ the directional cluster set of f at x in the direction θ is equal to the total cluster set of f at x. Wilczyński (1) [7] introduced the notion of a qualitative cluster set and obtained analogues of several results previously known for cluster sets and essential cluster sets [3]-[5]. We supplement [7] by showing in Section 3 that for functions having the Baire property the qualitative cluster set analogue of Collingwood's result is valid. A collection of four examples is given in Section 4 to indicate the sharpness of this result. We begin by establishing notation and definitions.

2. Notation and definitions. The functions considered here are those from the open upper half plane H into the Riemann sphere W. A countable basis for the topology on W will be denoted by $\mathscr{B}$. The point $(x, 0)$ on the real axis R will be denoted by x. For $x \in R$, $\theta \in (0, \pi)$ and $r > 0$ we let

$$K(x, r) = \{z \in H: |z-x| < r\}$$

and

$$L(x, \theta, r) = K(x, r) \cap \{z: \arg(z-x) = \theta\}.$$

Definition 1. Let $E \subset H$. Set

$$E_{\mathrm{II}} = \{x \in R: K(x, r) \cap E \text{ is of second category for each } r > 0\}$$

and

$$E_{\mathrm{I}} = R - E_{\mathrm{II}}.$$

For $\theta \in (0, \pi)$, set

$$E_{\mathrm{II}}(\theta) = \{x \in R: L(x, \theta, r) \cap E \text{ is of second category for each } r > 0\}$$

(1) The authors wish to thank Professor Frederick Bagemihl for calling their attention to the work of W. Wilczyński.

and

$$E_{\mathrm{I}}(\theta) = R - E_{\mathrm{II}}(\theta).$$

For $x \in R$, set

$$(E_{\mathrm{II}})(x) = \{\theta \in (0, \pi)\colon\ x \in E_{\mathrm{II}}(\theta)\}$$

and

$$(E_{\mathrm{I}})(x) = (0, \pi) - (E_{\mathrm{II}})(x).$$

Definition 2. Let $f\colon H \to W$ and $x \in R$. The *qualitative cluster set of f at x, $C_{\mathrm{q}}(f, x)$,* is the set of all points $y \in W$ such that $x \in f^{-1}(B)_{\mathrm{II}}$ for every $B \in \mathscr{B}$ with $y \in B$. For $\theta \in (0, \pi)$ the *qualitative directional cluster set of f at x in the direction θ, $C_{\mathrm{q}}(f, x, \theta)$,* is defined in the obvious analogous manner.

3. The analogue of Collingwood's result. In this section we state and prove a qualitative cluster set analogue (Theorem 2) of Collingwood's result mentioned in Section 1. An elementary property of arbitrary functions is pointed out in Theorem 1. Prior to the proof of each theorem we establish a lemma dealing with a related property of sets in H.

LEMMA 1. *If $E \subset H$ and*

$$S = \{x\colon\ (E_{\mathrm{II}})(x) \text{ is of second category}\},$$

then $S \subset E_{\mathrm{II}}$.

Proof. Let $x \in S$ and for each positive r set

$$A_r(x) = \{\theta\colon\ L(x, \theta, r) \cap E \text{ is of second category}\}.$$

Then

$$(E_{\mathrm{II}})(x) \subseteq A_r(x).$$

Consequently, $A_r(x)$ is of second category. Then an application of the Kuratowski-Ulam theorem (see [6], p. 56) guarantees that $K(x, r) \cap E$ is of second category, and hence $x \in E_{\mathrm{II}}$.

THEOREM 1. *Let $f\colon H \to W$ be arbitrary, and for each $x \in R$ set*

$$A(x) = \{\theta\colon\ C_{\mathrm{q}}(f, x, \theta) \subset C_{\mathrm{q}}(f, x)\}.$$

Then $A(x)$ is residual for each $x \in R$.

Proof. Suppose the contrary; that is, suppose that there is an $x \in R$ such that $A'(x) \equiv (0, \pi) - A(x)$ is of second category. Let $\theta \in A'(x)$ and choose a such that

$$a \in C_{\mathrm{q}}(f, x, \theta) - C_{\mathrm{q}}(f, x).$$

Then there is a $B \in \mathscr{B}$ such that $a \in B$, but $B \cap C_{\mathrm{q}}(f, x) = \emptyset$. Hence, $x \notin f^{-1}(B)_{\mathrm{II}}$, and $\theta \in \big(f^{-1}(B)_{\mathrm{II}}\big)(x)$. Since $A'(x)$ is of second category and $\mathscr{B}$

is countable, there is a $B \in \mathscr{B}$ such that $\Lambda'(x) \cap (f^{-1}(B)_{\mathrm{II}})(x)$ is of second category and $x \notin f^{-1}(B)_{\mathrm{II}}$. This, however, contradicts Lemma 1, and the theorem is proved.

LEMMA 2. *Let $E \subset H$ have the Baire property. The set*

$$A = \{x\colon (E_{\mathrm{I}})(x) \text{ is of second category}\} \cap E_{\mathrm{II}}$$

is of first category.

Proof. As the first case, suppose that E is open. Let $x \in A$ and for each $n = 1, 2, \ldots$ set

$$\Phi_n(x) = \{\theta\colon E \cap L(x, \theta, n^{-1}) \text{ is of first category}\}.$$

Notice that $E \cap L(x, \theta, n^{-1}) = \emptyset$ for $\theta \in \Phi_n(x)$. Clearly, each $\Phi_n(x)$ is closed, and

$$(E_{\mathrm{I}})(x) = \bigcup_{n=1}^{\infty} \Phi_n(x).$$

Since $(E_{\mathrm{I}})(x)$ is of second category, there is an n_0 such that $\Phi_{n_0}(x)$ contains an interval. Let a be a rational number in that interval. Then $x \in E_{\mathrm{II}} \cap E_{\mathrm{I}}(a)$. However, by applying Lemma 3 in [7] it is seen that the set $E_{\mathrm{II}} \cap \left[\bigcup E_{\mathrm{I}}(\beta)\right]$ (where the union is taken over all rational numbers β in $(0, \pi)$) is of first category. Consequently, A is of first category if E is open.

Now consider the general case where $E = G \Delta Q$ (Δ denotes the symmetric difference), G being open and Q of first category. We then have $E_{\mathrm{II}} = G_{\mathrm{II}}$. In view of this and the above case, we note that in order to show that A is of first category it suffices to show that the set

$$B = A \cap \{x\colon (G_{\mathrm{I}})(x) \text{ is of first category}\}$$

is of first category.

In fact, we can show that B is empty. For suppose $x \in B$. For each $n = 1, 2, \ldots$ set

$$\Phi_n(x) = \{\theta\colon E \cap L(x, \theta, n^{-1}) \text{ is of first category}\}.$$

Choose n_0 such that $\Phi_{n_0}(x)$ is of second category. We have

$$\Phi_{n_0}(x) \subset \{\theta\colon (G-Q) \cap L(x, \theta, n^{-1}) \text{ is of first category}\}.$$

However, $\{\theta\colon G \cap L(x, \theta, n^{-1}) \text{ is of first category}\}$ is a first category set as it is contained in $(G_{\mathrm{I}})(x)$. In light of the Kuratowski-Ulam theorem (see [6], p. 56) the two sets

$$\{\theta\colon G \cap L(x, \theta, n^{-1}) \text{ is of first category}\}$$

and

$$\{\theta\colon (G-Q) \cap L(x, \theta, n^{-1}) \text{ is of first category}\}$$

differ by at most a first category set. Hence the latter is also of first category, implying that $\Phi_{n_0}(x)$ is of first category. This contradiction shows that B is empty, and the proof is complete.

Now we can readily prove our main result.

THEOREM 2. *Let $f\colon H \to W$ have the Baire property and for each $x \in R$ set*

$$\Gamma(x) = \{\theta\colon C_q(f, x) = C_q(f, x, \theta)\}.$$

Then $\Gamma(x)$ is residual for nearly every $x \in R$.

Proof. Let

$$C = \{x\colon \Gamma^*(x) \text{ is of second category}\},$$

where

$$\Gamma^*(x) = \{\theta\colon C_q(f, x) \not\subset C_q(f, x, \theta)\}.$$

Let $x \in C$ and $\gamma \in \Gamma^*(x)$. Then there is a $B \in \mathscr{B}$ such that

$$x \in f^{-1}(B)_{\mathrm{II}} \quad \text{and} \quad x \in f^{-1}(B)_{\mathrm{I}}(\gamma).$$

Since $\Gamma^*(x)$ is of second category, there is a $B \in \mathscr{B}$ such that $x \in f^{-1}(B)_{\mathrm{II}}$ and $\big(f^{-1}(B)_{\mathrm{I}}\big)(x)$ is of second category. If C is of second category, then there is a $B \in \mathscr{B}$ such that

$$\big\{x\colon \big(f^{-1}(B)_{\mathrm{I}}\big)(x) \text{ is of second category}\big\} \cap f^{-1}(B)_{\mathrm{II}}$$

is of second category. This contradicts Lemma 2 and, consequently, C must be of first category.

This together with Theorem 1 completes the proof.

4. Examples. In this section, we present four examples which complement the theorems of Section 3.

Example 1. *There exists a measurable $f\colon H \to R$ such that, for every real x, $C_q(f, x) \not\subset C_q(f, x, \theta)$ for every direction θ.*

Proof. Theorem 15.5 in [4], p. 57, guarantees that there is a set E which is of second category in the unit square $I^2 = [0, 1] \times [0, 1]$ and which has the property that no three points of E are collinear. Let Q be a measure zero set which is residual in I^2 and let $K = E \cap Q$. Then K is a measure zero set which is of second category in I^2.

We extend K to a measure zero set which is of second category in every disc by placing a copy of K in every rational subsquare of H as follows.

Let

$$K_{r_1 r_2 r_3} = \{r_1 z + (r_2 + r_3 i)\colon z \in K\} \qquad (i = \sqrt{-1})$$

for each triple (r_1, r_2, r_3) of rational numbers for which r_1 and r_3 are positive. Then set

$$S = \bigcup K_{r_1 r_2 r_3},$$

where the union is taken over rational triples (r_1, r_2, r_3) of the appropriate form. Since S is the denumerable union of sets of measure zero, S is of measure zero, and S is of second category in every disc in H. Further, S contains at most denumerably many points on any given line.

Now we define $f\colon H \to \{0, 1\}$ by

$$f(z) = \begin{cases} 0 & \text{if } x \in S, \\ 1 & \text{if } x \notin S. \end{cases}$$

Then, for every real number x we have $0 \in C_q(f, x)$, but, as S is at most denumerable on every line, $C_q(f, x, \theta) = \{1\}$ for every direction θ, and the first example is established.

Example 2. *There is a continuous* $f\colon H \to [0, 1]$ *such that, for every real* x, $C_q(f, x) \not\subset C_q(f, x, \theta)$ *for almost every direction* θ.

Proof. Let $\{r_1, r_2, \ldots\}$ be an enumeration of the rational numbers. For each $n = 1, 2, \ldots$ we construct two discs, $D_{n,1}$ and $D_{n,2}$, centered at $(r_n, 1/n)$, with radii $[1/n]\tan(\pi/2^{n+2})$ and $[1/2n]\tan(\pi/2^{n+2})$, respectively. We take $D_{n,1}$ to be closed, and $D_{n,2}$ to be open. Let

$$A = \bigcup_{n=1}^{\infty} D_{n,2} \quad \text{and} \quad B = H - \bigcup_{n=1}^{\infty} D_{n,1}.$$

Then each of A and B is a closed subset of H and, consequently, there is a continuous function $f\colon H \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Let D denote a closed disc contained in H, and let $x \in R$. We let $\sigma(x, D)$ denote that sector in H which has x as an initial point and whose bounding rays are tangent to D. The directions of the rays bounding $\sigma(x, D)$ are denoted by $\theta_1(x, D)$ and $\theta_2(x, D)$, where $0 < \theta_1(x, D) < \theta_2(x, D) < \pi$.

If $x \in R$, then

$$[\theta_2(x, D_{n,1}) - \theta_1(x, D_{n,1})] < \frac{\pi}{2^{n+2}},$$

and hence

$$\Theta^1(x) = (0, \pi) - \bigcup_{n=1}^{\infty} \big(\theta_1(x, D_{n,1}), \theta_2(x, D_{n,1})\big)$$

has measure at least $\pi/2$. Further, if $\theta \in \Theta^1(x)$, then the ray $L(x, \theta)$ at x in the direction θ misses every disc $D_{n,1}$ $(n = 1, 2, \ldots)$, and hence the cluster set of f at x in the direction θ is $\{1\}$. Let

$$\Theta^2(x) = (0, \pi) - \bigcup_{n=2}^{\infty} \big(\theta_1(x, D_{n,1}), \theta_2(x, D_{n,1})\big).$$

Then $\Theta^2(x)$ has measure at least $3\pi/4$, and if $\theta \in \Theta^2(x)$, then $L(x, \theta)$ misses every disc $D_{n,1}$ ($n = 2, 3, \ldots$), and again the cluster set at x in the direction θ is $\{1\}$. Inductively, then, we set

$$\Theta^k(x) = (0, \pi) - \bigcup_{n=k}^{\infty} \big(\theta_1(x, D_{n,1}), \theta_2(x, D_{n,1})\big)$$

and note that both the measure of $\Theta^k(x)$ is at least $(2^k-1)\pi/2^k$ and that if $\theta \in \Theta^k(x)$, then the cluster set of f in the direction θ is $\{1\}$. It follows that if

$$\Theta(x) = \bigcup_{n=1}^{\infty} \Theta^k(x),$$

then both the measure of $\Theta(x)$ is π, and that if $\theta \in \Theta(x)$, then the cluster set of f at x in the direction θ is $\{1\}$. Further, every semidisc $K(x, r)$ contains a disc of the form $D_{n,2}$ and an open subset of B. Hence, $\{0, 1\} \subset C_q(f, x)$, thus establishing the result.

Remark. If we use $C(f, x, \theta)$ to denote the ordinary cluster set of f at x in the direction θ, then Example 2 shows:

There is a continuous function f: $H \to [0, 1]$ such that, for every real x, $C_q(f, x) \not\subset C(f, x, \theta)$ for almost every direction θ.

This demonstrates a striking difference in the behavior of essential cluster sets and qualitative cluster sets of continuous functions as evidenced by the following theorem from [1] where the subscript e indicates an essential cluster set.

THEOREM BEH. *Let f: $H \to W$ be continuous and for each $x \in R$ set*

$$\Theta(x) = \{\theta\colon\ C_e(f, x) \subset C(f, x, \theta)\}.$$

Then $\Theta(x)$ is both residual and of full Lebesgue measure for almost every and nearly every $x \in R$.

Example 3. *There is a continuous f: $H \to R$ such that, for each point x of a set K of positive measure,*

$$\{\theta\colon\ C_q(f, x) \not\subset C_q(f, x, \theta)\}$$

is of second category.

Proof. Let K be a closed nowhere dense subset of R with positive measure. Let $\{I_n\colon n = 1, 2, \ldots\}$ be an enumeration of the intervals contiguous to K, and let T_n be the closed equilateral triangular region in H whose base is I_n. At each point $x \in K$ let W_x denote the closed unbounded wedge in H

having x as an initial point and bounding rays in the directions $2\pi/5$ and $3\pi/5$. If

$$z \in \bigcup_{n=1}^{\infty} T_n$$

is the vertex of an equilateral triangle having its base on the x-axis, then that base is contained in $\bigcup_{n=1}^{\infty} I_n$. It follows, then, that $\bigcup_{n=1}^{\infty} T_n$ and $\bigcup_{x\in K} W_x$ are mutually exclusive closed subsets of H.

There is a continuous $f\colon H \to [0,1]$ such that

$$f(z) = \begin{cases} 0 & \text{for } z \in \bigcup_{n=1}^{\infty} T_n, \\ 1 & \text{for } z \in \bigcup_{x\in K} W_x. \end{cases}$$

As K is nowhere dense and each set T_n contains an open subset, $1 \in C_q(f, x)$ for every $x \in K$. However, for $\theta \in [2\pi/5, 3\pi/5]$ and $x \in K$ we have $C_q(f, x, \theta) = \{0\}$, and the example is established.

Example 4. *Let $\Gamma(x)$ be as in Theorem 2. Then there is a continuous $f\colon H \to W$ such that $\bigcap_{x\in Q} \Gamma(x)$ is of first category for each residual set Q in R.*

This result follows directly from the following statement ([2], Theorem 2) wherein $\Theta(x)$ is as in Theorem BEH.

THEOREM HEB. *There is a continuous $f\colon H \to W$ such that $\bigcap_{x\in Q} \Theta(x)$ is of first category for each residual set Q in R.*

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