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ON A CERTAIN CLASS OF RIEMANNIAN HOMOGENEOUS SPACES

BY

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DEDICATED TO
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WITH AN APPRECIATION
OF THE WORK AND INFLUENCE

A well-known theorem due to I. M. Gelfand, A. Selberg and others says that the algebra of invariant differential operators on a globally symmetric Riemannian space is commutative. There are many important applications of this theorem in several branches of mathematics such as the theory of spherical functions, the theory of numbers, mathematical physics, etc. In this short report we have a try at discussion of the converse of this theorem, namely to what extent a Riemannian homogeneous space satisfying commutativity condition of the algebra of invariant differential operators is close to a symmetric Riemannian space, at least locally. It is one of the main results of the present paper that under such condition there hold identities

$$\nabla_k R_{ij} + \nabla_i R_{jk} + \nabla_j R_{ik} = 0, \tag{1}$$

where R_{ij} and $\nabla_k R_{ij}$ are components of Ricci tensor and components of covariant differential of Ricci tensor, respectively, though the meaning of equation (1) from either a point of view of Riemannian geometry or of hermitian geometry is yet unknown.

The paper is organized as follows. It contains a series of five theorems. The first three study basic properties of invariant differential operators in homogeneous spaces and we take the liberty to omit their proofs and detailed explanation and to quote standard references. The last two are, trifling as they seem to be, communications by quick delivery of our originals. Details of the proofs are postponed to a forthcoming paper [5].

Throughout the paper we use notation and terminology which belong to the so called "Ricci calculus". For instance, Einstein's convention of dummy indices is frequently employed.

Let M be a differentiable manifold and let $C_c^\infty(M)$ denote the algebra of smooth functions (real valued) whose carriers are compact. A linear endomorphism D of $C_c^\infty(M)$ into itself is called a *differential operator on M* if the following two conditions are satisfied:

(i) It is a continuous mapping with respect to the pseudo-topology of $C_c^\infty(M)$ in the sense of L. Schwartz.

(ii) It has a local character, that is, the carrier of Df is a subset of the carrier of f, where f is any element of $C_c^\infty(M)$.

This is a global definition of a differential operator on a differentiable manifold, but if we restrict the domain to a sufficiently small neighborhood of an arbitrary point of M, any differential operator in the above-mentioned sense will coincide with a differential operator in the classical sense. Assertion of the following theorem explains it briefly and precisely:

THEOREM A. *Let D be any differential operator on M and let f be any differentiable function having a compact carrier in a certain coordinate neighborhood U in M. Then the function Df can be expressed in U as a finite linear combination of differentials (with smooth functions as coefficients) as follows:*

$$Df = \sum_p \sum_{i_1 \dots i_p} a_{i_1 \dots i_p} \frac{\partial^p f}{\partial x^{i_1} \dots \partial x^{i_p}}. \tag{2}$$

For the sake of brevity a function on U and its coordinate expression are identified in (2).

Theorem A is a simple consequence of a well known-theorem due to L. Schwartz that any distribution whose carrier consists of only one point can be regarded as a linear combination (with real coefficients) of Dirac δ-function and its succesive derivatives (derivatives in the sense of the theory of distribution). We refer for details to the fundamental and instructive paper [2] where Lichnerowicz gives also an improvement of theorem A for the use of a student of tensor calculus. The following theorem, which makes a brief summary of it, is our starting point in the present paper:

THEOREM B. *Any differential operator D on a manifold with a linear connection can be expressed as a linear combination of covariant derivatives as*

$$Df = \sum_{p=1}^{k} T^{i_1 \dots i_p} \nabla_{i_1} \dots \nabla_{i_p} f, \tag{3}$$

where each coefficient $T^{i_1 \dots i_p}$ is a contravariant component of a symmetric tensor field and k is a degree of D.

Let G/H be a Riemannian homogeneous space, that is, there exists a Riemannian metric tensor which is an invariant of the action of a connected Lie group G upon its quotient space G/H, where H is a closed subgroup of G and the action of G is assumed to be effective.

A differential operator D on G/H is called *invariant* if

$$D(f\cdot g)\cdot g^{-1} = Df, \tag{4}$$

where f is any smooth function and $f\cdot g$ denotes the function defined by composite product of f with the action of $g \epsilon G$ as a diffeomorphism of G/H. As is well known, the Laplacean operator defined by $\Delta f = g^{ij}\nabla_i \nabla_j f$ with respect to the invariant Riemannian metric g^{ij} is easily seen to be an invariant operator in the above-mentioned sense on any Riemannian homogeneous space. More generally, any invariant differential operator can be characterized by the invariantness of coefficients of a contravariant tensor in (3) as follows:

THEOREM C. *Any invariant differential operator on a Riemannian homogeneous space G/H can be expressed as a linear combination of the form*

$$Df = \sum T^{i_1\cdots i_p}\nabla_{i_1}\dots\nabla_{i_p} f, \tag{5}$$

where $T^{i_1\cdots i_p}$ are components of a smooth symmetric tensor field of order p, which is an invariant with respect to the action of the group G, i.e., the Lie derivative of which with respect to any infinitesimal transformation belonging to G vanishes.

For the proof of this theorem we also refer to [2].

It is a known theorem of E. Cartan that any invariant tensor in a symmetric homogeneous space is a parallel tensor with respect to the canonical connection. Nevertheless, in an arbitrary Riemannian homogeneous space the Levi-Cività connection need not be canonical in the sense of Nomizu [4], and so invariant tensor need not be a parallel one. From now on we will be concerned mainly with a rather special class of Riemannian homogeneous space characterized by the commutativity of the algebra of invariant differential operator. For the sake of brevity, we call such a space "with the condition c".

THEOREM D. *In a Riemannian homogeneous space with the condition* c *any invariant symmetric tensor field $T_{i_1\dots i_p}$ must satisfy the following two conditions*:

$$\nabla_{i_{p+1}}T_{i_1\dots i_p}+\nabla_{i_1}T_{i_2\dots i_p i_{p+1}}+\dots+\nabla_{i_p}T_{i_{p+1}i_1\dots i_{p-1}} = 0, \tag{6}$$

$$\nabla^a T_{i_1\dots i_{p-1}a} = 0. \tag{7}$$

Outline of the proof. Take any invariant differential operator D of order p and express it with the aid of theorem C as $T^{i_1\cdots i_p}\nabla_{i_1}\dots\nabla_{i_p}$.

On the other hand, since there is an invariant differential operator called Laplacean, our hypothesis of satisfying commutativity condition yields the identity

$$(DΔ - ΔD)f = 0. \tag{8}$$

In order to avoid vain efforts of tremendous complexity of calculations, we assume that D is a differential operator of order 2. In this case, identity (8) can be rewritten as

$$\begin{aligned}(DΔ - ΔD)f = T^{kl} g^{ij} (\nabla_l \nabla_k \nabla_j \nabla_i f - \nabla_j \nabla_i \nabla_l \nabla_k f) - \\ - g^{ij} (\nabla_j \nabla_i T^{kl}) \nabla_l \nabla_k f - 2g^{ij} (\nabla_j T^{kl})(\nabla_i \nabla_l \nabla_k) f.\end{aligned} \tag{9}$$

By the application of Ricci formula in tensor calculus, we can easily show that the first and the second members of the right-hand side of (9) contain derivatives of f of order at most 2, while the third member has ∇T^{kl} as the coefficient of the derivative of third order. From (8) and the arbitrariness of f we obtain

$$\nabla_k T_{ij} + \nabla_i T_{jk} + \nabla_j T_{ik} = 0. \tag{10}$$

Remarks. (i) Identity (1) is a special case of (6) for D being a differential operator $R^{ij} \nabla_i \nabla_j f$. (ii) In addition to (1) there are many identities involving derivatives of components of the curvature tensor in a space with the condition c. For example

$$\nabla_k (R_{iabc} R_j^{abc}) + \nabla_i (R_{jabc} R_k^{abc}) + \nabla_j (R_{iabc} R_k^{abc}) = 0, \tag{11}$$

$$\nabla_k (R_{ia} R^a{}_j) + \nabla_i (R_{ja} R^a{}_k) + \nabla_j (R_{ia} R^a{}_k) = 0. \tag{12}$$

(11) is a special case of (6) for D being a differential operator $R^{iabc} R^j_{abc} \nabla_i \nabla_j$, while (12) is that in the case of $R^i{}_a R^{ja} \nabla_i \nabla_j$.

Having learned these identities, we propose the following open problem:

Is a Riemannian homogeneous space with the condition c *a space with parallel Ricci tensor?* (**P 807**)

The assumption "Riemannian" is indispensable, for Helgason [1] studied a certain homogeneous space whose ring of invariant differential operators has the structure of a polynomial ring and so is non-Riemannian.

Let us study another application of commutativity.

THEOREM E. *In a compact Riemannian homogeneous space with the condition* c, *any harmonic vector is a parallel vector.*

Yano and Bochner ([6], cf. also [3]) have shown that the inner product of a Killing vector and a harmonic vector is constant. Thus we conclude that a harmonic vector is invariant, and so from (6) it is also a Killing vector.

There are also many other applications in the tensor calculus.

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