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SEMIGROUPS OF OPERATORS COMMUTING WITH TRANSLATIONS

BY

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1. Introduction. Let G denote a compact Abelian group with the character group X. We denote by $C(G)$ the Banach space of all continuous complex-valued functions on G with the uniform norm. We write λ for Haar measure on G.

Let $L_p(G), 1 \leqslant p < \infty$, have the usual meaning. To avoid unnecessary repetition, we shall write $\mathscr{U} = \mathscr{U}(G)$ for an arbitrary, but fixed, member of the set

$$\{C(G);\ L_p(G)\colon\ 1 \leqslant p < \infty\}.$$

Given $A \subset \mathscr{U}$, we denote by $\hat{A}$ the set of all Fourier transforms $\hat{f}$ of $f \in A$. A complex-valued function φ on X is called an (A, B)-*multiplier* ([1] and [2]) if $\varphi\hat{f} \in \hat{B}$ for each $f \in A$, where A and B are subsets of $\mathscr{U}$. A collection $\mathscr{G} = \{T(\xi)\colon\ \xi > 0\}$ of bounded linear operators on $\mathscr{U}$ is said to be a *semigroup* of operators on $\mathscr{U}$ if

$$T(\xi_1+\xi_2) = T(\xi_1)T(\xi_2) \quad \text{for all } \xi_1, \xi_2 > 0.$$

A comprehensive account of semigroups of operators on Banach spaces can be found in Hille and Phillips [3], where all undefined terms used in the present paper in connection with such semigroups are explained.

The following two theorems are our main results.

1.1. THEOREM. *Let $\mathscr{G} = \{T(\xi)\colon\ \xi > 0\}$ be a semigroup of bounded linear operators on $\mathscr{U}$. Suppose that, for each $\xi > 0$, the operator $T(\xi)$ commutes with translations. Then $\mathscr{G}$ defines a semigroup $\mathscr{M} = \{E_\xi\colon\ \xi > 0\}$ of $(\mathscr{U}, \mathscr{U})$-multipliers such that*

(i) *for each $\xi > 0$, $E_\xi\hat{f} = (T(\xi)f)\hat{}$ for all $f \in \mathscr{U}$ and*

(ii) *$E_{\xi_1+\xi_2}(\sigma) = E_{\xi_1}(\sigma)E_{\xi_2}(\sigma)$, $\xi_1, \xi_2 > 0$ and $\sigma \in X$.*

If, moreover, $T(\xi)$ is weakly measurable, then there exist a subset X_0 of X and a mapping $\varphi\colon\ \sigma \to \varphi_\sigma$ of X_0 into the field K of complex numbers *such that*

$$E_\xi(\sigma) = \begin{cases} e^{\varphi_\sigma \xi} & \textit{if } \sigma \in X_0, \\ 0 & \textit{if } \sigma \notin X_0 \end{cases}$$

for each $\xi > 0$.

Suppose now that X_0 is a fixed subset of X and let $\varphi\colon \sigma \to \varphi_\sigma$ be a mapping of X_0 into K. Put

$$E_\xi(\sigma) = \begin{cases} e^{\varphi_\sigma \xi} & \text{if } \sigma \in X_0, \\ 0 & \text{if } \sigma \notin X_0. \end{cases}$$

Assume that E_ξ as defined here is a $(\mathscr{U}, \mathscr{U})$-multiplier. Then we have the following

1.2. Theorem. *For each $\xi > 0$, define a mapping $T(\xi)$ of $\mathscr{U}$ into itself by*

$$\big(T(\xi)f\big)^\wedge = E_\xi \hat{f}, \quad f \in \mathscr{U}.$$

Then:

(i) $\mathscr{G} = \{T(\xi)\colon \xi > 0\}$ *defines a semigroup of bounded linear operators on $\mathscr{U}$, the elements of which commute with translations and which is continuous in the strong operator topology for $\xi > 0$;*

(ii) *for each $f \in D(A_0)$ and $\sigma \notin X_0$ we have $\hat{f}(\sigma) = 0$, where A_0 denotes the infinitesimal generator of $\mathscr{G}$ and $D(A_0)$ is the domain of A_0; moreover, φ is a $\big(D(A_0), \mathscr{U}\big)$-multiplier since $(A_0 f)^\wedge = \varphi\hat{f}$ for all $f \in D(A_0)$;*

(iii) *$\mathscr{G}$ being of class (A), $X_0 = X$ and $D(A) = \{f \in \mathscr{U}\colon \varphi\hat{f} \in \hat{\mathscr{U}}\}$ (i.e. φ is a $\big(D(A), \mathscr{U}\big)$-multiplier) and, moreover, $(Af)^\wedge = \varphi\hat{f}$ for all $f \in D(A)$.*

Theorems 1.1 and 1.2 are generalizations to compact Abelian groups of results by Hille and Phillips, proved for the circle group ([3], Theorems 20.3.1 and 20.3.2). As in the circle group situation, our results for $\mathscr{U} = L_2(G)$ provide much more information than those stated in 1.1 and 1.2. Here, for semigroups $\mathscr{G} = \{T(\xi)\colon \xi > 0\}$ commuting with translations, the weak continuity of $T(\xi)$ implies the strong convergence at the origin and, therefore, the strong continuity for all $\xi > 0$. Furthermore, as in the circle group case, given an arbitrary closed subset F in the left half-plane of K, we are able to construct a semigroup $\mathscr{G}$ of class (C_0) on $L_2(G)$ such that the spectrum of the infinitesimal generator of $\mathscr{G}$ is precisely F.

Our proofs are an adaptation of Hille's proofs in the circle group situation.

2. Proofs of main results. For the sake of clarity, we shall throughout this paper write χ_σ for a continuous character of G when it is considered as a function on G, and σ for the same character when it is considered as an element of the character group X.

2.1. Proof of Theorem 1.1. Since the bounded linear operator $T(\xi)$, $\xi > 0$, on $\mathscr{U}$ commutes with translations, there exists a $(\mathscr{U}, \mathscr{U})$-

multiplier E_ξ such that

$$(T(\xi)f)\hat{} = E_\xi \hat{f} \quad \text{for all } f \in \mathscr{U}$$

(cf. [4], p. 3-5).

Moreover, a routine argument shows that E_ξ is unique. This proves the first part of the theorem and the second part follows immediately from the semigroup property of $\{T(\xi): \xi > 0\}$.

Suppose now that $T(\xi)$ is weakly measurable. Then for each $\psi \in \mathscr{U}^*$, where $\mathscr{U}^*$ denotes the space of all continuous linear functionals on $\mathscr{U}$, and for each $f \in \mathscr{U}$, $\xi \to \psi(T(\xi)f)$ is Lebesgue measurable. In particular, if for each σ we define $\psi_\sigma \in \mathscr{U}^*$ by

$$\psi_\sigma(f) = (f)\hat{}\,(\sigma), \quad f \in \mathscr{U},$$

then $\xi \to \psi_\sigma(T(\xi)\chi_\sigma) = E_\xi(\sigma)\hat{\chi}_\sigma(\sigma)$ is measurable. It follows that, for each σ, $E_\xi(\sigma)$ is measurable. Since also, $E_{\xi_1+\xi_2}(\sigma) = E_{\xi_1}(\sigma)E_{\xi_2}(\sigma)$, it follows from corollary to Theorem 4.17.3 of [3] that, for each σ, either $E_\xi(\sigma)$ is identically zero or $E_\xi(\sigma) = e^{\varphi_\sigma \xi}$ for some complex number φ_σ. Now set $X_0 = \{\sigma \in X: E_\xi(\sigma) \neq 0\}$ and the proof is complete.

2.2. Proof of Theorem 1.2. (i) That $T(\xi)$ is a bounded linear operator for each $\xi > 0$ follows from (35.2) of [2]. The semigroup property and the fact that the operators commute with translations are immediate from the definition of $T(\xi)$.

We shall prove that $T(\xi)$ is continuous in the strong operator topology for $\xi > 0$. First, suppose that $t \in I(G)$, the set of all finite complex linear combinations of continuous characters of G. Thus t is of the form

$$t = a_1\chi_{\sigma_1} + a_2\chi_{\sigma_2} + \dots + a_n\chi_{\sigma_n}.$$

The orthogonality of $I(G)$ implies that $T(\xi)t$ is defined by

$$(T(\xi)t)(x) = a_1 e^{\varphi_{\sigma_1}\xi}\chi_{\sigma_1}(x) + \dots + a_n e^{\varphi_{\sigma_n}\xi}\chi_{\sigma_n}(x), \qquad x \in G.$$

We then have

$$\begin{aligned} &\|T(\xi)t - T(\xi_0)t\| \\ &\quad = \|(a_1 e^{\varphi_{\sigma_1}\xi}\chi_{\sigma_1} - a_1 e^{\varphi_{\sigma_1}\xi_0}\chi_{\sigma_1}) + \dots + (a_n e^{\varphi_{\sigma_n}\xi}\chi_{\sigma_n} - a_n e^{\varphi_{\sigma_n}\xi_0}\chi_{\sigma_n})\| \\ &\quad \leqslant |a_1|\,|e^{\varphi_{\sigma_1}\xi} - e^{\varphi_{\sigma_1}\xi_0}| + \dots + |a_n|\,|e^{\varphi_{\sigma_n}\xi} - e^{\varphi_{\sigma_n}\xi_0}| \to 0 \end{aligned}$$

as $\xi \to \xi_0$. Suppose now that f is arbitrary in $\mathscr{U}$ and let $\varepsilon > 0$ be given. Then there exists $t \in I(G)$ such that $\|f - t\| < \varepsilon$. Since

$$\|T(\xi)f - T(\xi)t\| \leqslant \|T(\xi)\| \cdot \|f - t\| \quad \text{for each } \xi > 0,$$

$T(\xi)f$ is strongly measurable by [3], Theorem 3.5.4. Hence, by [3], Theorem 10.2.3, $T(\xi)$ is continuous in the strong operator topology for $\xi > 0$. This completes the proof of (i).

(ii) Let $f \in D(A_0)$. Then there exists $g = A_0 f \in \mathscr{U}$ such that

$$\lim_{\eta \to 0^+} \frac{1}{\eta} [T(\eta)f - f] = g$$

in the norm topology. For each σ,

$$\frac{1}{\eta} \big[(T(\eta)f)^{\wedge}(\sigma) - \hat{f}(\sigma)\big] \to \hat{g}(\sigma), \quad \text{i.e.} \quad \frac{1}{\eta} [E_\eta(\sigma) - 1]\hat{f}(\sigma) \to \hat{g}(\sigma) \qquad \text{as } \eta \to 0^+.$$

As a consequence of the definition of E_η, it follows that $\hat{f}(\sigma) = 0$ for $\sigma \notin X_0$ and

$$(A_0 f)^{\wedge}(\sigma) = \varphi_\sigma \hat{f}(\sigma) \qquad \text{for all } \sigma \in X_0,$$

which proves (ii).

(iii) Suppose that $\{T(\xi): \xi > 0\}$ is of class (A) with the infinitesimal generator $A = \bar{A}_0$, the smallest closed extension of its infinitesimal operator A_0. Then $\mathscr{U}_0 = \{T(\xi)f: f \in \mathscr{U}, \xi > 0\}$ and $D(A_0)$ are dense in $\mathscr{U}$. Suppose there exists $\sigma_0 \in X$ such that $\sigma_0 \notin X_0$; choose $f \in \mathscr{U}$ such that $\hat{f}(\sigma_0) \neq 0$. Then given $\varepsilon > 0$, there exists an $f' \in D(A_0)$ such that $\|f' - f\| < \varepsilon$. Then

$$|\hat{f}'(\sigma_0) - \hat{f}(\sigma_0)| \leqslant \|f' - f\| < \varepsilon$$

which implies that $\hat{f}(\sigma_0) = 0$, a contradiction. Hence X_0 is the whole of X.

Finally, let w_0 be the type of the semigroup $\{T(\xi): \xi > 0\}$. For λ with $\mathrm{Re}(\lambda) > w_0$, let $R(\lambda; A)$ denote the resolvent of the infinitesimal generator A of $\{T(\xi): \xi > 0\}$. Then there exists a $w_1 > w_0$ such that

$$R(\lambda; A)f = \int_0^\infty e^{-\lambda\xi} T(\xi) f d\xi, \qquad f \in \mathscr{U}_0, \mathrm{Re}(\lambda) > w_1.$$

Since, for each $\sigma \in X$, the mapping $f \to \hat{f}(\sigma)$ is a bounded linear functional on $\mathscr{U}$, we have, for all $f \in \mathscr{U}_0$,

$$\begin{aligned}(R(\lambda; A)f)^{\wedge}(\sigma) &= \int_0^\infty e^{-\lambda\xi} (T(\xi)f)^{\wedge}(\sigma) d\xi \\ &= \int_0^\infty e^{-\lambda\xi} e^{\varphi_\sigma \xi} \hat{f}(\sigma) d\xi = (\lambda - \varphi_\sigma)^{-1} \hat{f}(\sigma) \qquad \text{for each } \sigma \in X.\end{aligned}$$

Since $\mathscr{U}_0$ is dense in $\mathscr{U}$, we have

$$(1) \qquad (R(\lambda; A)f)^{\wedge}(\sigma) = (\lambda - \varphi_\sigma)^{-1} \hat{f}(\sigma) \qquad \text{for all } f \in \mathscr{U}, \mathrm{Re}(\lambda) > w_1.$$

Let $\lambda > w_1$ be fixed and suppose that $f \in D(A)$. Then there exists a $g \in \mathscr{U}$ such that $f = R(\lambda; A)g$, and we have, for each $\sigma \in X$,

$$\begin{aligned}(Af)^{\wedge}(\sigma) &= [\lambda R(\lambda; A)g - g]^{\wedge}(\sigma) \\ &= \lambda(\lambda - \varphi_\sigma)^{-1} \hat{g}(\sigma) - \hat{g}(\sigma) \\ &= \varphi_\sigma (\lambda - \varphi_\sigma)^{-1} \hat{g}(\sigma) = \varphi_\sigma \hat{f}(\sigma).\end{aligned}$$

Thus if $f \in D(A)$, then $\varphi\hat{f} \in \hat{\mathcal{U}}$. Conversely, suppose that f is an element of $\mathcal{U}$ such that $\varphi\hat{f} \in \hat{\mathcal{U}}$. This means that there exists an $h \in \mathcal{U}$ such that $\varphi_\sigma \hat{f}(\sigma) = \hat{h}(\sigma)$ for all $\sigma \in X$. Then $g = \lambda f - h \in \mathcal{U}$ and, for all $\sigma \in X$,

$$\begin{aligned}[R(\lambda;\ A)g]\hat{\ }(\sigma) &= (\lambda - \varphi_\sigma)^{-1}\hat{g}(\sigma) \\ &= (\lambda - \varphi_\sigma)^{-1}\big(\lambda\hat{f}(\sigma) - \varphi_\sigma \hat{f}(\sigma)\big) = \hat{f}(\sigma).\end{aligned}$$

Hence $R(\lambda;\ A)g = f$ and $f \in D(A)$. This completes the proof of the theorem.

3. The $L_2(G)$ case.

3.1. Theorem. *Let $\mathcal{G} = \{T(\xi)\colon \xi > 0\}$ be a semigroup of bounded linear operators on $L_2(G)$. Suppose that the operator $T(\xi)$ commutes with translations and is weakly measurable for $\xi > 0$. Then*

(i) $\big(T(\xi)f\big)\hat{\ } = E_\xi \hat{f}$ *for all* $f \in L_2(G)$, $\xi > 0$, *and*

$$E_\xi(\sigma) = \begin{cases} e^{\varphi_\sigma \xi} & \text{if } \sigma \in X_0, \\ 0 & \text{if } \sigma \notin X_0; \end{cases}$$

(ii) $\|T(\xi)\|_2 = e^{\nu\xi}$, *where* $\nu = \sup_{\sigma \in X_0} \mathrm{Re}(\varphi_\sigma)$;

(iii) *if $T(0)$ is given by*

$$\big(T(0)f\big)\hat{\ }(\sigma) = \begin{cases} \hat{f}(\sigma) & \text{if } \sigma \in X_0, \\ 0 & \text{if } \sigma \notin X_0, \end{cases}$$

then $T(\xi)$ is strongly continuous for $\xi \geqslant 0$ and if $X_0 = X$, then $\mathcal{G}$ is of class (C_0), i. e.

$$\lim_{\xi \to 0^+} T(\xi)f = f \quad \text{for all } f \in L_2(G).$$

Proof. (i) has been established in Theorem 1.1.

(ii) By (i),

$$\big(T(\xi)f\big)\hat{\ } = E_\xi \hat{f} \quad \text{for all } f \in L_2(G).$$

Obviously, E_ξ is bounded on X for each $\xi > 0$ and $f \to \hat{f}$ is a norm preserving isomorphism of $L_2(G)$ onto $l_2(X)$. It follows that

$$\|T(\xi)\|_2 = \sup_{\sigma \in X} |E_\xi(\sigma)| = \sup_{\sigma \in X_0} |e^{\varphi_\sigma \xi}| = e^{\nu\xi}, \quad \text{where } \nu = \sup_{\sigma \in X_0} \mathrm{Re}(\varphi_\sigma).$$

(iii) We already know from Theorem 1.1 that $T(\xi)$ is strongly continuous for $\xi > 0$. Now, for each $f \in L_2(G)$,

$$\begin{aligned}\|[T(\xi) - T(0)]f\|_2^2 &= \big\|\big([T(\xi) - T(0)]f\big)\hat{\ }\big\|_2^2 \\ &= \sum_{\sigma \in X_0} |e^{\varphi_\sigma \xi} - 1|^2 |\hat{f}(\sigma)|^2 \to 0 \quad \text{as } \xi \to 0^+.\end{aligned}$$

Thus $T(\xi)$ is strongly continuous at $\xi = 0$. Now if $X_0 = X$, then

$$\big(T(0)f\big)\hat{}\,(\sigma) = \hat{f}(\sigma) \quad \text{for all } f \in L_2(G) \text{ and } \sigma \in X.$$

Therefore $T(0)$ is the identity operator; so

$$\lim_{\xi \to 0^+} T(\xi) f = f \quad \text{for all } f \in L_2(G),$$

and the theorem is proved completely.

Finally we prove the following

3.2. THEOREM. *Let $F = \{\lambda\} \subset K$ be any closed set such that* $\mathrm{Re}(\lambda) \leqslant a < \infty$. *Then there exists a semigroup of class (C_0) on $L_2(G)$ whose elements commute with translations and such that the spectrum $\sigma(A)$ of its infinitesimal generator A is F.*

Proof. Set $X_0 = X$. Choose φ on X into K such that F is the closure of $\{\varphi_\sigma\colon \sigma \in X\}$ and define $T(\xi)$, $\xi > 0$, by 3.1(i). Then, by Theorems 1.2 and 3.1, $\{T(\xi)\colon \xi > 0\}$ is of class (C_0). Moreover, by 2.2(1), the resolvent $R(\lambda; A)$ of its infinitesimal generator A is given by

$$\big(R(\lambda; A)f\big)\hat{}\,(\sigma) = (\lambda - \varphi_\sigma)^{-1} \hat{f}(\sigma), \qquad \sigma \in X.$$

Consequently, $\lambda \notin \sigma(A)$ if and only if $\inf_{\sigma \in X} |\lambda - \varphi_\sigma| > 0$. Hence $\sigma(A) = F$, which proves the theorem.

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