COLLOQUIUM MATHEMATICUM

VOL. XXV 1972 FASC. 1

ON TRANSITIVE OPERATIONS IN ABSTRACT ALGEBRAS

BY

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An n-ary operation f is said to be *transitive* if for each pair $i, j \in \{1, \ldots, n\}$ there exists a permutation $p_1, \ldots, p_n$ of numbers $1, \ldots, n$ such that $p_i = j$ and

$$f(x_1, \ldots, x_n) = f(x_{p_1}, \ldots, x_{p_n}).$$

In other words, f is called transitive if the group of symmetry of f is transitive.

In this paper we give a generalization of some results, proved earlier by E. Marczewski and K. Urbanik, concerning symmetrical and quasi-symmetrical operations. It is also a continuation of investigations of cyclic operations by J. Płonka.

For any abstract algebra $\mathfrak{A}$ we denote by $S(\mathfrak{A})$ the set of all integers $n \geqslant 2$ for which there exists an n-ary algebraic operation in $\mathfrak{A}$ depending on every variable.

Marczewski proved ([1], Corollary 3) the following theorem:

If there are no algebraic constants in algebra $\mathfrak{A} = (A; f)$, where f is k-ary quasi-symmetrical operation in $\mathfrak{A}$ $(k \geqslant 2)$, then the set $S(\mathfrak{A})$ contains arithmetical progression $k+j(k-1)$, $(j = 0, 1, 2, \ldots)$.

Urbanik proved ([5], Theorem 1), under the same assumptions, that the set $S(\mathfrak{A})$ contains arithmetical progression $k+2+j(k-1)$ $(j = 0, 1, 2, \ldots)$, and that if, moreover, $S(\mathfrak{A})$ is an arithmetical progression itself, then $S(\mathfrak{A}) = \{2, 3, \ldots\}$, $\{3, 4, \ldots\}$ or $\{3, 5, \ldots\}$ ([5], Theorem 2).

Płonka [3] proved that if f is cyclic, then $S(\mathfrak{A}) \supset \{1, 3, \ldots\}$. We shall prove that if f is a k-ary transitive operation (to be quasi-symmetrical or cyclic are stronger conditions for f than to be transitive), then $S(A; f)$ is the union of an arbitrary set of even integers and of the set of all odd integers. In our construction f will be even symmetrical, thus assuming quasi-symmetry or symmetry does not give a new thesis. If a transitive operation f depends on an even number of variables, then $S(\mathfrak{A}) = \{2, 3, \ldots\}$ or $\{3, 4, \ldots\}$.

Our conclusions from Lemma 1 can be obtained also from results of Płonka [4].

We assume throughout the paper that no algebra has constant operations. Put $F^0(x) = x$, $F^{k+1}(x) = f(F^k(x), \ldots, F^k(x))$ for any operation $f(x_1, \ldots, x_n)$. $|A|$ will denote the number of elements in A.

LEMMA 1. *Let $f(x_1, \ldots, x_n)$ $(n \geqslant 3)$ be a transitive operation in an algebra $\mathfrak{A}$. If $h(x_1, \ldots, x_{k-1}, F^r(x_k))$ is an algebraic operation depending on all variables $x_1, \ldots, x_k$, then there exists an index $i \in \{1, \ldots, n\}$ such that*

$$h\big(x_1, \ldots, x_{k-1}, f(F^{r-1}(x), \ldots, F^{r-1}(x), F^{r-1}(y), F^{r-1}(z), \ldots, F^{r-1}(z))\big),$$

where $F^{r-1}(y)$ stands in the i-th place in f, depends on all $k+2$ variables $x_1, \ldots, x_{k-1}, x, y, z$ $(r = 1, 2, \ldots)$.

Proof. Examine the operation

$$g(x_1, \ldots, x_{k-1}, x, y, z) = h\big(x_1, \ldots, x_{k-1}, f(F^{r-k}(x), F^{r-1}(y), F^{r-1}(z), \ldots, F^{r-1}(z))\big).$$

Putting $x = y = z$ into g we see that g depends on $x_1, \ldots, x_{k-1}$ and on at least one of x, y, z. Put

$$g_j(x_1, \ldots, x_{k-1}, x, z) = h\big(x_1, \ldots, x_{k-1}, f(F^{r-1}(x), \ldots, F^{r-1}(x), F^{r-1}(z), \ldots, F^{r-1}(z))\big),$$

where $F^{r-1}(x)$ is repeated j times and $F^{r-1}(z)$ is repeated $n-j$ times $(j = 1, \ldots, n)$.

(a) If g does not depend on x, then $g(x_1, \ldots, x_{k-1}, x, z, z) = g_1(x_1, \ldots \ldots, x_{k-1}, x, z)$ also does not depend on x, but g_1 must depend on z. From the transitivity of f we infer that $f(t, \ldots, t, s) = f(s, t, \ldots, t)$. Thus

$$g_{n-1}(x_1, \ldots, x_{k-1}, x, z) = g(x_1, \ldots, x_{k-1}, x, x, z) = g(x_1, \ldots, x_{k-1}, z, x, x)$$

depends on x and does not depend on z. Therefore there exists an index $i \in \{1, \ldots, n\}$ such that

$$(1) \qquad g_{i-1}(x_1, \ldots, x_{k-1}, x, z) = g'(x_1, \ldots, x_{k-1}, z)$$

does not depend on x and

$$(2) \qquad g_i(x_1, \ldots, x_{k-1}, x, z) = g''(x_1, \ldots, x_{k-1}, x)$$

does not depend on z. It follows that the operation

$$u(x_1, \ldots, x_{k-1}, x, y, z) = h\big(x_1, \ldots, x_{k-1}, f(F^{r-1}(x), \ldots, F^{r-1}(x), F^{r-1}(y), F^{r-1}(z), \ldots, F^{r-1}(z))\big),$$

where $F^{r-1}(y)$ stands in the i-th place in f, depends on $x_1, \ldots, x_{k-1}, x, y, z$, because, putting $y = z$ into u, we get $u(x_1, \ldots, x_{k-1}, x, z, z) = g'(x_1, \ldots \ldots, x_{k-1}, z)$ by (1), and (putting $y = x$) $u(x_1, \ldots, x_{k-1}, x, x, z) = g''(x_1, \ldots \ldots, x_{k-1}, x)$ by (2).

By the transitivity of f we have $f(t, s, t, \ldots, t) = f(s, t, \ldots, t)$ and so we can proceed in the same way if g does not depend on y.

(b) If now g does not depend on z, but depend on x and y, then $g_2(x_1, \ldots, x_{k-1}, x, z) = g(x_1, \ldots, x_{k-1}, x, x, z)$ does not depend on z. On the other hand, putting $z = y$ in g, we infer that the operation $g_1(x_1, \ldots \ldots, x_{k-1}, x, y)$ depends on x and, by the transitivity of f, the operation $g_n(x_1, \ldots, x_{k-1}, x, z)$ depends on z. Therefore there exists an $i \in \{2, \ldots, n\}$ such that

(3) $$g_{i-1}(x_1, \ldots, x_{k-1}, x, z) \text{ does not depend on } z,$$

(4) $$g_i(x_1, \ldots, x_{k-1}, x, z) \text{ depends on } z.$$

Putting $F^{r-1}(y)$ in the i-th place in f, we get an operation

$$u(x_1, \ldots, x_{k-1}, x, y, z)$$
$$= h\big(x_1, \ldots, x_{k-1}, f\big(F^{r-1}(x), \ldots, F^{r-1}(x), F^{r-1}(y), F^{r-1}(z), \ldots, F^{r-1}(z)\big)\big)$$

depending on $x_1, \ldots, x_{k-1}, x, y, z$, because, by (3),

$$u(x_1, \ldots, x_{k-1}, x, z, z) = g_{i-1}(x_1, \ldots, x_{k-1}, x, z)$$

does not depend on z, and, by (4),

$$u(x_1, \ldots, x_{k-1}, x, x, z) = g_i(x_1, \ldots, x_{k-1}, x, z)$$

depends on z.

Remark. If r can be arbitrarily large, then one can show by induction that there are in $\mathfrak{A}$ operations depending on $k+2i$ variables $(i = 1, 2, \ldots)$.

COROLLARY. *If $f(x_1, \ldots, x_n)$, $n \geqslant 3$, is a transitive operation in $\mathfrak{A}$, then $S(\mathfrak{A}) \supset \{3, 5, \ldots\}$.*

Proof. $F^{r+1}(x) = F\big(F^r(x)\big)$ depends on x for every integer r. Lemma 1 with $F^{r+1}(x)$ in the place of h yields the thesis.

LEMMA 2. *Let $f(x_1, \ldots, x_n)$ be a transitive operation in $\mathfrak{A}$. If f, after identification of all variables to two, depends on one of them only, then there exists an $i \in \{2, \ldots, n\}$ such that the operation*

$$g(x, y, z) = f(x, \ldots, x, y, z, \ldots, z),$$

where y stands in the i-th place, depends on x, y, z.

Moreover, if $f(x, y, \ldots, y) = F(x)$, then g satisfies

(5) $$g(x, y, y) = g(y, x, y) = g(y, y, x) = G(x) = F(x)$$

and if $f(x, y, \ldots, y) = F(y)$, *then* g *satisfies*

$$g(x, y, y) = g(y, x, y) = g(y, y, x) = G(y) = F(y). \tag{6}$$

Proof. Put $f_j(x, z) = f(x_1, \ldots, x_n)$, where $x_1 = x_2 = \ldots = x_j = x$, $x_{j+1} = \ldots = x_n = z$. By assumption, $f_1(x, z) = f(x, z, \ldots, z)$ depends only on x or only on z.

Let $f_1(x, z) = F(x)$ do not depend on z. By the transitivity of f we have $f_n(x, z) = F(z)$. Thus there exists an $i \in \{2, \ldots, n\}$ such that

$$f_{i-1}(x, z) = F(x) \quad \text{and} \quad f_i(x, z) = F(z).$$

Put $g(x, y, z) = f(x_1, \ldots, x_n)$, where $x_1 = \ldots = x_{i-1} = x$, $x_i = y$, $x_{i+1} = \ldots = x_n = z$. We have

$$g(x, z, z) = f_{i-1}(x, z) = F(x), \quad g(x, x, z) = f_i(x, z) = F(z)$$

and, by the transitivity of f,

$$g(x, y, x) = f(x, \ldots, x, y, x, \ldots, x) = F(y);$$

thus g satisfies (5).

If $f(x, y, \ldots, y) = F(y)$, we can in the same way obtain an operation satisfying (6). Formulas (5) or (6) give then directly the desired dependence on variables.

LEMMA 3. *If* $g(x_1, x_2, x_3)$ *is an operation satysfying, for any* $x, y \in A$, *the equalities*

$$g(x, x, y) = g(x, y, x) = g(y, x, x) = G(x),$$

then the operation

$$u(x_1, x_2, x_3, x_4) = g\big(x_1, g(x_2, x_3, x_1), G^r(x_4)\big)$$

depends on x_1, x_2, x_3, x_4 *for each* $r = 1, 2, \ldots$

Proof. Putting $x_2 = x_3 = x$ we get

$$u(x_1, x, x, x_4) = g\big(x_1, g(x, x, x_1), G^r(x_4)\big) = g\big(x_1, G(x), G^r(x_4)\big)$$

and so u depends on x_1 and x_4. Let now $x_2 = x_1$. Since

$$g\big(x_1, g(x_1, x_3, x_1), G^r(x_4)\big) = g\big(x_1, G(x_1), G^r(x_4)\big) \neq g\big(x_1, G(x_3), G^r(x_4)\big),$$

u depends on x_2. The same for x_3.

LEMMA 4. *Let* $f(x_1, \ldots, x_{2n})$, $n \geqslant 2$, *be a transitive operation in* $\mathfrak{A}$. *If* f, *after identification of all variables to two, depends on one of them only, then* $S(\mathfrak{A}) \supset \{3, 4, \ldots\}$.

Proof. By assumption, $f(x, y, \ldots, y) = F(y)$ or $f(x, y, \ldots, y) = F(x)$. Since in the first case Lemmas 2, 3 and 1 (see also Corollary) yield the thesis, consider the second case: $f(x, y, \ldots, y) = F(x)$. By Lemma 2, there is in A an operation $g(x,y,z)$ satisfying (5). From the transitivity of f we have

$$f(y,\ldots,y,x,y,\ldots,y) = F(x), \tag{7}$$

where x stands in any place.

Examine operations $f_i(x,z) = f(x,\ldots,x,z,\ldots,z)$ $(i = 1,\ldots,2n)$, where the first i variables in f are equal to x and the remaining ones to z.

There are three possibilities:

(a) $f_{2i+1}(x,z) = F(x)$, $f_{2i}(x,z) = F(z)$ for $i = 0,\ldots,n-1$,

(b) there exists a $j \in \{2,\ldots,2n-1\}$ with $f_{j-1}(x,z) = f_j(x,z) = F(x)$,

(c) there exists a $j \in \{3,\ldots,2n\}$ with $f_{j-1}(x,z) = f_j(x,z) = F(z)$.

In the case (a) we have $f_{2n-1}(x,z) = F(x)$. On the other hand, by (7), $f_{2n-1}(x,z) = f(x,\ldots,x,z) = F(z)$. Thus case (a) is impossible.

In the case (b) put $h(x,y,z) = f(x,\ldots,x,y,z,\ldots,z)$, where the first $j-1$ variables in f are equal to x. Operation h satisfies

$$\begin{gathered} h(x,y,y) = f_{j-1}(x,y) = F(x), \qquad h(x,y,x) = F(y), \\ h(x,x,z) = f_j(x,z) = F(x). \end{gathered} \tag{8}$$

In view of (8), $F^r(h(x,y,z))$ and $h(F^r(x),F^r(y),F^r(z))$ depend on x,y,z $(r = 1,2,\ldots)$. Examine the following operation u:

$$u(x_1,x_2,x_3,x_4) = g\big(x_1,x_2,h(x_1,x_3,F^r(x_4))\big).$$

If $x_1 = x_2$, then, by (5),

$$u(x_2,x_2,x_3,x_4) = g\big(x_2,x_2,h(x_2,x_3,F^r(x_4))\big) = F\big(h(x_2,x_3,F^r(x_4))\big). \tag{9}$$

The right-hand side of (9) depends on x_2,x_3,x_4, and thus u depends on x_3 and x_4. Inserting $x_2 = h(x_1,x_3,F^r(x_4))$ in u, we infer, by (5) and (9), that u depends on x_2, because

$$u\big(x_1,h(x_1,x_3,F^r(x_4)),x_3,x_4\big) = F(x_1) \neq F\big(h(x_1,x_3,F^r(x_4))\big).$$

Now put $x_1 = x_3$ into u. We obtain, by (8), the equality

$$u(x_3,x_2,x_3,x_4) = g\big(x_3,x_2,h(x_3,x_3,F^r(x_4))\big) = g\big(x_3,x_2,F(x_3)\big). \tag{10}$$

Operation on the right-hand side of (10) does not depend on x_4. On the other hand, according to (9), $u(x_2,x_2,x_3,x_4)$ depends on x_4, and so u depends also on x_1. Lemma 1 applied successively to operations g and u in the place of h gives thesis. This finishes the proof in the case (b).

In the case (c) put $h_1(x,y,z) = f(x,\ldots,x,y,z,\ldots,z)$, where the first $j-1$ variables of f are equal to x. By assumption of (c) and (7), operation h_1 has the properties $h_1(x,y,y) = f_{j-1}(x,y) = F(y)$, $h_1(x,y,x) = F(y)$, $h_1(x,x,z) = f_j(x,z) = F(z)$. Since the operation $h(x,y,z) = h_1(z,y,x)$ satisfies (8), the case (c) can be reduced to that of (b).

LEMMA 5. *Let* $f(x_1, \ldots, x_n)$ $(n \geqslant 3)$ *be a transitive operation in* $\mathfrak{A}$. *If* $S(\mathfrak{A}) \neq \{2, 3, \ldots\}$, *then there exists an integer* r *such that* $f\big(F^r(x_1), \ldots \ldots, F^r(x_n)\big)$ *depends, after identification of all variables* $x_1, \ldots, x_n$ *to two, on one of them only.*

Proof. If for any r from $f\big(F^r(x_1), \ldots, F^r(x_n)\big)$ we could obtain, by the identification of variables, an operation $h\big(F^r(x), F^r(y)\big)$ depending on two variables, then Lemma 1 would give $S(\mathfrak{A}) \supset \{2, 4, \ldots\}$ and so, by Corollary, $S(\mathfrak{A}) \supset \{2, 3, \ldots\}$. A contradiction.

THEOREM 1. *If in an algebra* $\mathfrak{A}$ *without constant operations there is a transitive operation depending on an even number of variables, then* $S(\mathfrak{A}) \supset \{3, 4, \ldots\}$.

Proof. For binary operations transitivity is equivalent to symmetry. Płonka [2] proved that if in an algebra $\mathfrak{A}$ there is a binary symmetrical operation, then $S(\mathfrak{A}) \supset \{2, 3, \ldots\}$. If there is in $\mathfrak{A}$ a transitive operation $f(x_1, \ldots, x_{2n})$, $n \geqslant 2$, then Lemma 4, applied either to f or to $f\big(F^r(x_1), \ldots \ldots, F^r(x_{2n})\big)$ from Lemma 5, gives the thesis.

THEOREM 2. *For any set* T *of even integers there exists an algebra* $\mathfrak{A} = (A; f)$ *such that the fundamental operation* f *is ternary symmetrical and* $S(\mathfrak{A})$ *is equal to the union of* T *and of the set of all odd integers. Moreover, for each* $k = 1, 2, \ldots$, *there is in* $\mathfrak{A}$ *a symmetrical operation depending on* $2k+1$ *variables.*

Proof. Take the set $A = \{a, b, a_1, a_2, \ldots, c_1, c_2, \ldots, 2, 4, \ldots\}$, where sets $\{a\}, \{b\}, \{a_1, \ldots\}, \{c_1, \ldots\}\ \{2, \ldots\}$ are pairwise disjoint and define on A the operation

$$g(x_1, x_2, x_3) = \begin{cases} b & \text{if } x_i = b \text{ for the odd number of } i\text{'s}, \\ a & \text{otherwise.} \end{cases}$$

It is easy to see that g is symmetrical, satisfies (5), and that for each $k = 2, 3, \ldots$ the operations

$$g_3(x_1, x_2, x_3) = g(x_3, x_2, x_1),$$

$$g_{2k+1}(x_1, \ldots, x_{2k+1}) = g\big(x_{2k+1}, x_{2k}, g_{2k-1}(x_1, \ldots, x_{2k-1})\big)$$

satisfy the condition

$$g_{2k+1}(x_1, \ldots, x_{2k+1}) = \begin{cases} b & \text{if } x_i = b \text{ for the odd number of } i\text{'s}, \\ a & \text{otherwise.} \end{cases}$$

These $g_{2k+1}(k = 1, 2, \ldots)$ are symmetrical and, of course, depend on $x_1, \ldots, x_{2k+1}$. Any non-trivial algebraic operation in $(A; g)$ is equal to g_{2k+1} for some k.

Now we shall define the operation f.

If $T = \emptyset$, put $f = g$. If $T = \{t_1, t_2, \ldots\}$ (finite or not), put (arguments in f can be placed in any order):

$$f(a_1, a_2, a_1) = 2, \quad f(a_5, a_4, a_3) = c_1, \quad f(a_3, a_6, c_1) = 4,$$

$$f(a_9, a_8, a_7) = c_2, \quad f(a_{11}, a_{10}, c_2) = c_3, \quad f(a_7, a_{12}, c_3) = 6, \quad \ldots,$$

$$f(a_{m+3}, a_{m+2}, a_{m+1}) = c_{n+1}, \quad f(a_{m+5}, a_{m+4}, c_{n+1}) = c_{n+2}, \quad \ldots,$$

$$f(a_{m+2k-1}, a_{m+2k-2}, c_{n+k-2}) = c_{n+k-1},$$

$$f(a_{m+1}, a_{m+2k}, c_{n+k-1}) = 2k, \quad \ldots,$$

where $m = m(k) = k(k-1)$, $n = n(k) = \frac{1}{2}(k-2)(k-1)$. In the remaining cases put $f(x_1, x_2, x_3) = g(x_1, x_2, x_3)$.

If $2k \notin T$, we delete the equalities from $f(\ldots) = 2(k-1)$ to $f(\ldots) = 2k$, but without the first of them.

Now define operations $f_{2k+1}(x_1, \ldots, x_{2k+1})$, algebraic in $\mathfrak{A} = (A; f)$, as follows:

$$f_3(x_1, x_2, x_3) = f(x_3, x_2, x_1),$$

$$f_{2k+1}(x_1, \ldots, x_{2k+1}) = f\big(x_{2k+1}, x_{2k}, f_{2k-1}(x_1, \ldots, x_{2k-1})\big) \quad (k = 2, 3, \ldots).$$

In other words, for $k = 1, 2, \ldots$, we have

$$f_{2k+1} = f\Big(x_{2k+1}, x_{2k}, f\big(x_{2k-1}, x_{2k-2}, f(\ldots, f(x_3, x_2, x_1))\ldots\big)\Big).$$

Put

$$f_{2k}(x_1, \ldots, x_{2k}) = f_{2k+1}(x_1, x_2, \ldots, x_{2k}, x_1).$$

In this way we obtain for $2k \in T$ operations f_{2k} satisfying

$$f_{2k}(a_{k(k-1)+1}, \ldots, a_{k(k+1)}) = 2k. \tag{11}$$

From the symmetry of f, variables x_{2i} and x_{2i+1} $(1 \leqslant i \leqslant k-1)$ commute in f_{2k} and these cycles generate the whole group of symmetry of f_{2k}.

Finally, observe that

$$f_{2k}(x_1, \ldots, x_{2k}) = g_{2k-1}(x_2, \ldots, x_{2k}) \quad \text{if } \{x_1, \ldots, x_{2k}\} \neq \{a_1, \ldots, a_{2k}\}. \tag{12}$$

Operations f_{2k} depend on $x_1, \ldots, x_{2k}$, because, putting in the left-hand side of (11) $x_i = a$ instead of $i \in \{1, \ldots, 2k\}$, we obtain on the right-hand side a instead of $2k$. Observe that any superposition of ternary operation f has an odd number of variables. If $h(x_1, \ldots, x_{2k})$ is an algebraic operation in $\mathfrak{A} = (A; f)$, then, in any formula determining h, some variables are repeated an even number of times (formally, f is superposed with a trivial operation). If in h there is an even number of repetitions, different from that in f_{2k}, then, by the definition of f, h does not depend on repeated variables. Thus any algebraic operation depending on $2k$ variables is equal to f_{2k}. Finally, observe that

$$F\big(f(x_1, x_2, x_3)\big) = f\big(F(x_1), F(x_2), F(x_3)\big) = g(x_1, x_2, x_3).$$

Thus g is an algebraic operation in $\mathfrak{A}$ and operations g_{2k+1} are the wanted symmetrical operations depending on $2k+1$ variables for each $k = 1, 2, \ldots$ This completes the proof.

If f is a k-ary, where k is an even, cyclic operation, then f generates a binary symmetrical operation. We shall show that there exists a two-element idempotent algebra with a transitive operation of six variables and without binary non-trivial operations.

Example. Put $\mathfrak{A} = (0, 1; f)$, where

$$f(x_1, \ldots, x_6) = x \quad \text{if } |\{i\colon x_i = x\}| > 3$$

and

$$\begin{aligned} f(x, x, x, y, y, y) &= f(x, x, y, x, y, y) = f(x, y, x, y, y, x) \\ &= f(y, x, y, x, y, x) = f(y, x, x, y, x, y) \\ &= f(x, y, y, x, x, y) = f(x, y, y, y, x, x) \\ &= f(y, y, x, x, x, y) = f(y, x, y, y, x, x) \\ &= f(y, y, x, x, y, x) = x. \end{aligned}$$

Operation f is transitive. The group of symmetry of f is generated by $(1, 2, 3)(4, 5, 6)$ and $(1, 2)(3, 4)5, 6$. Let h be a k-ary operation $(k > 2)$. For each subset P of $\{1, \ldots, k\}$ we define a binary operation h_P by

$$h_P(x, y) = f(x_1, \ldots, x_k),$$

where $x_i = x$ if $i \in P$, and $x_i = y$ if $i \notin P$. Observe that $f_P(x, y) = e_j^2(x, y)$, $j = 1, 2$, where, for any n, trivial operations e_j^n are defined by $e_j^n(x_1, \ldots, x_n) = x_j$ $(j = 1, \ldots, n)$. Let h_j $(j = 1, \ldots, n)$ be k-ary operations on A and h be an n-ary operation on A. Let $h_{jP}(x, y)$ $(j = 1, \ldots, n)$ be equal to a trivial operation for each $P \subset \{1, \ldots, k\}$ and let also $h_Q(x, y)$ be equal to a trivial operation for any $Q \subset \{1, \ldots, n\}$. Examine the superposition

$$u(x_1, \ldots, x_k) = h(h_1, \ldots, h_n)(x_1, \ldots, x_k).$$

Observe that

$$u_P(x, y) = h(h_{lP}, \ldots, h_{nP})(x, y) = h_Q(x, y)$$

for some $Q \subset \{1, \ldots, n\}$ is a trivial operation. Thus in our algebra $\mathfrak{A} = (A; f)$ each binary operation is trivial and $2 \notin S(\mathfrak{A})$.

Finally, observe that in this example the group of symmetry of f is doubly transitive and f is quasi-symmetrical.

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Reçu par la Rédaction le 23. 10. 1970