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SOME PROPERTIES OF MEASURABLE SETS

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By R^n we shall mean an n-dimensional Euclidean space, $|X|$ will denote the n-dimensional Lebesgue measure of $X \subset R^n$, and a point $p \in X$ will be called a *density point* of X (in the sense of measure) if

$$\lim_{\varrho \to 0} \frac{|K(p, \varrho) \cap X|}{|K(p, \varrho)|} = 1,$$

where $K(p, \varrho)$ is the ball with centre p and radius ϱ. As is easy to observe, the ball $K(p, \varrho)$ can be replaced here by the cube $C(p, \varrho)$ with centre p and edge ϱ.

The set of all density points of X will be denoted by X^0.

p is said to be a *boundary point* of X (in the sense of measure) if it does not belong to $X^0 \cup (X')^0$, where $X' = R^n \setminus X$.

The set of all boundary points of X will be denoted by $X^{\cdot}$. Clearly,

$$X^{\cdot} = [X^0 \cup (X')^0]'.$$

LEMMA 1. *Let $\{M_k\}$ be a sequence of n-dimensional cubes with edges $d_k \to 0$ and let X be a subset of R^n. If $p \in X^0$ and $p \in M_k$ for $k = 1, 2, \ldots$, then*

$$\lim_{k \to \infty} \frac{|M_k \cap X|}{|M_k|} = 1.$$

Proof. Without loss of generality we may assume $n = 3$. Suppose the lemma is not true. This means that for some sequence of cubes $\{M_k\}$ there is

$$\lim_{k \to \infty} \frac{|M_k \cap X|}{|M_k|} = 1 - \delta < 1,$$

i.e.

$$\lim_{k \to \infty} \frac{|M_k \cap X'|}{|M_k|} = \delta > 0.$$

Let $\tilde{M}_k$ be the least cube with centre p emboding M_k. The edge of $\tilde{M}_k$ does not then exceed $2d_k$ and so $|\tilde{M}_k| \leqslant 8|M_k|$. Consequently,

$$\frac{|\tilde{M}_k \cap X|}{|\tilde{M}_k|} = 1 - \frac{|\tilde{M}_k \cap X'|}{|\tilde{M}_k|} \leqslant 1 - \frac{|M_k \cap X'|}{8|M_k|},$$

whence, finally,

$$\overline{\lim_k} \frac{|\tilde{M}_k \cap X|}{|\tilde{M}_k|} \leqslant 1 - \frac{\delta}{8} < 1$$

contrary to hypothesis $p \in X^0$.

LEMMA 2. *Let $X \subset R^n$. If X is measurable, then X^0 and $X^{\cdot}$ are Borel.*

Indeed, if X is measurable, then $p \in X^0$ if and only if for each integer k there exists an integer r such that

$$\frac{|K(p, 1/r) \cap X|}{|K(p, 1/r)|} > 1 - \frac{1}{k},$$

and this equivalence implies that X^0 is a G_δ. Since by the same argument also $(X')^0$ is a G_δ, we see that $X^{\cdot}$ must be F_σ.

Now let A be an arbitrary measurable subset of R^n and let P be an $(n-1)$-dimensional hyperplane of R^n, $n \geqslant 2$. Denote by $\boldsymbol{V}$ the family of all straight lines L perpendicular to P and meeting both A^0 and $(A')^0$, i.e.

$$\boldsymbol{V} = \{L\colon L \perp P, L \cap A^0 \neq 0 \text{ and } L \cap (A')^0 \neq 0\},$$

and by V the union of all $L \in \boldsymbol{V}$, i.e. $V = \bigcup_{L \in \boldsymbol{V}} L$.

Main aim of the present paper is to prove that almost every (in a sense to be precised later on) straight line L of the family $\boldsymbol{V}$ meets $A^{\cdot}$.

Main lemma is

LEMMA 3. *If $\boldsymbol{W} \subset \boldsymbol{V}$ and $W \cap P$, where $W = \bigcup_{L \in \boldsymbol{W}} L$, is closed and has positive $(n-1)$-dimensional measure, then there exists a straight line $L \in \boldsymbol{W}$ such that $L \cap A^{\cdot} \neq 0$.*

Proof. Assume $n = 3$ and consider in R^3 a system of coordinates in which the plane Oxy coincides with P and all straight lines of the family $\boldsymbol{V}$ are parallel to Oz.

It suffices to construct a sequence of cubes $\{M_k\}$ satisfying the following conditions:

(a) $M_1 \supset M_2 \supset \ldots$ and $d(M_{k+1}) \leqslant \frac{1}{2} d(M_k)$, where $d(M)$ denotes the edge of the cube M,

(b) $$\frac{1}{4} \leqslant \frac{|M_k \cap A|}{|M_k|} \leqslant \frac{3}{4},$$

(c) $$M_k \cap W \neq 0.$$

Indeed, since W is closed by hypothesis, we infer in view of (a) and (c) that the set $\bigcap M_k \cap W$ consists of one point, say p_0, and, by virtue of (b) and Lemma 1, p_0 is a boundary point of A.

Sequence $\{M_k\}$ will be defined inductively. For the purpose of construction we replace condition (c) by the following one, a little stronger than (c) itself:

(c′) For each k there exists a family $\boldsymbol{W}_k \subset \boldsymbol{W}$ such that $L \cap \operatorname{Int} M_k \cap A^0 \neq 0$ and $L \cap \operatorname{Int} M_k \cap (A')^0 \neq 0$ for each $L \in \boldsymbol{W}_k$, the set $W_k \cap P$ (where $W_k = \bigcup_{L \in \boldsymbol{W}_k} L$) is closed and has positive 2-dimensional measure.

1. Construction of M_1 and $\boldsymbol{W}_1$. Let $L_0 \in \boldsymbol{W}$ be a straight line each point of which is a density point of W (such a straight line surely exists, since in view of hypothesis $|P \cap W| > 0$ the set $P \cap W$ contains a density point and each straight line meeting that point can be taken as L_0).

Let $M(d, z_0)$ denote the cube with the edge of length d, centre lying in L_0 and lower face lying in the plane $z = z_0$.

Consider two points, $p_1 = (x_0, y_0, z_1) \in L_0 \cap A^0$ and $p_2 = (x_0, y_0, z_2) \in L_0 \cap (A')^0$. In view of the defintion of L_0 there exist $p_1 \in (A \cap W)^0$ and $p_2 \in (A' \cap W)^0$.

Put $\varepsilon_0 = 1/10$. By virtue of Lemma 1 there exist then $d_1 > 0$ and $d_2 > 0$ such that

(i) if $0 < d \leqslant d_1$, then

$$\frac{|M(d, z_1) \cap A \cap W|}{|M(d, z_1)|} > 1 - \varepsilon_0,$$

(ii) if $0 < d \leqslant d_2$, then

$$\frac{|M(d, z_2) \cap A \cap W|}{|M(d, z_2)|} < \varepsilon_0.$$

Assume $z_1 < z_2$ and put $d_0 = \min(d_1, d_2)$ and $M_z = M(d_0, z)$. It is easy to check that the function

$$\varphi(z) = \frac{|M_z \cap A \cap W|}{|M_z|}$$

is continuous.

In view of (i) and (ii) there is $\varphi(z_1) > 1 - \varepsilon_0$ and $\varphi(z_2) < \varepsilon_0$. Hence and from the inequality $\varepsilon_0 = 1/10 < 1/2$ we infer that there exist points z' and z'' such that $z' < z''$, $z'' - z' < d_0$, $\varphi(z') = 1/2$ and $\frac{1}{2} - \varepsilon_0 < \varphi(z'') < \frac{1}{2}$ (note that one can take the greatest value in the set $\varphi^{-1}(\frac{1}{2})$ as z').

Denote by $\boldsymbol{W}_z$ the subfamily of $\boldsymbol{W}$ consisting of all straight lines L such that $L \cap \operatorname{Int} M_z \cap A^0 \neq 0$ and $L \cap \operatorname{Int} M_z \cap (A')^0 \neq 0$. Let W_z be the union of straight lines of $\boldsymbol{W}_z$. The set W_z is then a cylinder the base of which (on P) is a common part of projection of $\operatorname{Int} \boldsymbol{W}_z \cap A^0$

and Int $W_z \cap (A')^0$. By virtue of Lemma 2 the two sets are Borel; therefore W_z is analytic, hence measurable ([1]).

Equality $W_z = 0$ means that for each straight line L passing through Int M_z either (α) each point of $L \cap \text{Int}\, M_z$ belongs to $A^0 \cup A^{\cdot}$ or (β) to $(A')^0 \cup A^{\cdot}$. Thus if $|\xi_1 - \xi_2| < d_0$ and $W_{\xi_1} = W_{\xi_2} = 0$, then the set of straight lines satisfying (α) is the same for ξ_1 and ξ_2. Since the sets A and A' differ from $A^0 \cup A^{\cdot}$ and $(A')^0 \cup A^{\cdot}$, respectively, for sets of measure 0 only, we infer in view of the definition of φ that there is $\varphi(\xi_1) = \varphi(\xi_2)$. Now if $|W_z \cap P| = 0$, then alternative "(α) or (β)" holds for almost every perpendicular line passing through Int M_z, and so, reasoning as before, we infer that if $|\xi_1 - \xi_2| < d_0$ and $|W_{\xi_1} \cap P| = |W_{\xi_2} \cap P| = 0$, then $\varphi(\xi_1) = \varphi(\xi_2)$.

Hence and from $\varphi(z') \neq \varphi(z'')$ it follows that $|W_{z_0} \cap P| > 0$ for $z_0 = z'$ or $z_0 = z''$, and so the set $W_{z_0} \cap P$ contains a closed subset F of positive measure.

Putting now

$$M_1 = M_{z_0} \quad \text{and} \quad W_1 = \{L\colon L \| Oz \text{ and } L \cap F \neq 0\}$$

we easily see that M_1 and W_1 satisfy (c′) and the first inequality of (b).

Here is a proof of the second inequality of (b):

$$\frac{|M_z \cap A|}{|M_z|} = \frac{|M_z \cap A \cap W|}{|M_z|} + \frac{|M_z \cap A \cap W'|}{|M_z|}$$
$$= \varphi(z) + \frac{|M_z \cap W'|}{|M_z|} < \varphi(z') + \varepsilon_0 \leqslant \tfrac{1}{2} + \varepsilon_0,$$

where $z = z'$ or $z = z''$ and the last but one inequality follows by (c).

2. Construction of M_{k+1} and W_{k+1}. This construction does not differ essentially from that of M_1 and W_1 except that condition (a) needs a little more attention.

Assume inductive hypothesis that a cube M_k and a family W_k satisfying (a), (b) and (c′) have been constructed, and denote by L_0, similarly as in section 1, a straight line from W_k each point of which is a density point of W_k. This straight line contains points p_1 and p_2 such that $p_1 \in L_0 \cap A^0 \cap \text{Int}\, M_k$ and $p_2 \in L_0 \cap (A')^0 \cap \text{Int}\, M_k$. Choose numbers d_1 and d_2 as in section 1, but modify the definition of d_0 by putting

$$d_0 = \min\big(d_1, d_2, \tfrac{1}{2}d(M_k), \varrho(p_1, \text{Bd}\, M_k), \varrho(p_2, \text{Bd}\, M_k)\big),$$

where Bd M_k is the boundary of the cube M_k and $\varrho(p, A) = \inf\limits_{a \in A} \varrho(p, a)$. This definition guarantees that (a) holds.

([1]) C. Kuratowski, *Topologie I*, Warszawa 1958, p. 391.

Now until the end the construction runs as in section 1. Hence the proof of the existence of a sequence of cubes $\{M_k\}$ satisfying (a), (b) and (c′) (and so, a fortiori, (c)) has been completed.

THEOREM. *If A is a measurable subset of R^n, P an $(n-1)$-dimensional hyperplane of R^n, L a straight line perpendicular to P, and*

$$V = \{L\colon\ L \cap A^0 \neq 0,\, L \cap (A')^0 \neq 0 \text{ and } L \cap A^{\cdot} = 0\},$$

then $|V \cap P| = 0$, where $V = \bigcup_{L \in V} L$.

Proof. By virtue of Lemma 2 the sets A^0, $(A')^0$ and $A^{\cdot}$ are Borel. Since the set $V \cap P$ is a common part of projections of these sets into a hyperplane P, it is analytic, hence measurable. In view of Lemma 3 it cannot contain any closed subset of positive measure.

Remarks. As simple examples show, there need not to be $V \cap P = 0$.

Hyperplane P need not be perpendicular to straight lines from V; the proof analogous to that given above works for straight lines which are scew to P.

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