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CHOW RING OF PROJECTIVE NON-SINGULAR TORUS EMBEDDING

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Let X be an n-dimensional torus embedding over an algebraically closed field k, and let S be the corresponding finite rational polyhedral partial (f.r.p.p.) decomposition of $N_R \simeq R^n$ which will be also called the (*conical*) *complex* (see [7], Chapter I, § 1,2, for definitions). Our aim is to describe the Chow ring and the l-adic cohomology of X in terms of the complex S in the case of a projective non-singular X. Some results for the case of a complete X are also given.

1. A ring associated with the conical complex. Let us fix an f.r.p.p. decomposition S of R^n such that the corresponding variety $X = X_S$ is complete and non-singular. Let $a_1, a_2, \ldots, a_r$ be the one-dimensional faces of S (the letter a will be reserved for faces of dimension 1, and ω for faces of dimension n). Each semigroup $a_j \cap Z^n$ is spanned by an element $(a_{1j}, a_{2j}, \ldots, a_{nj})$ of Z^n. We attach to S the ring

$$(*) \qquad Z[U_1, U_2, \ldots, U_r]/I,$$

where U_j are variables corresponding to a_j, I is the homogeneous ideal generated by linear forms $\sum_{j=1}^{r} a_{ij} U_j$ $(i = 1, 2, \ldots, n)$ and monomials $U_{j_1} U_{j_2} \ldots U_{j_s}$ such that $(a_{j_1}, a_{j_2}, \ldots, a_{j_s})$ are minimal sequences spanning no face in S.

The ring $(*)$ will be denoted by $Z[U]/I$. It has a gradation such that the cosets of U_j are of degree 1. The Theorem from Section 2 states that $Z[U]/I$ is just the Chow ring of X.

Let us describe some properties of $Z[U]/I$.

For two cones $\sigma, \tau \in S$ we write $\sigma \prec \tau$ if σ is a face of τ, and we write σ non-$\prec$ τ if it is not. To each $\sigma \in S$, different from $\{0\}$, there corresponds a monomial of the form

$$P_\sigma = U_{j_1} U_{j_2} \ldots U_{j_s},$$

where $j_1, j_2, \ldots, j_s$ are such indices that $a_{j_1}, a_{j_2}, \ldots, a_{j_s}$ span σ. For example, $P_{a_j} = U_j$. The cosets of $U_1, U_2, \ldots, P_\sigma$ in $Z[U]/I$ will be denoted by $u_1, u_2, \ldots, p_\sigma$, respectively. For the cone $\{0\}$ we set $p_0 = 1$.

LEMMA 1 ("shifting away" lemma). *Let θ, σ, ω be elements of S such that $\theta \prec \sigma \prec \omega$ and $\theta \neq \sigma$. Then there exist cones $\sigma_l \in S$ and integers c_l such that*

$$\dim \sigma_l = \dim \sigma \quad \textit{for each } l,$$
$$p_\sigma = \sum_l c_l p_{\sigma_l}, \quad \sigma_l \,\text{non-}\!\prec \omega \quad \textit{for each } l.$$

Proof. It suffices to consider the case where $\dim \theta = \dim \sigma - 1$ and $\dim \omega = n$ (S is complete!). By changing the numeration we may assume that ω is spanned by $a_1, a_2, \ldots, a_n$, the face σ by $a_1, a_2, \ldots, a_s$, where $s \leqslant n$, and θ by $a_2, a_3, \ldots, a_s$. We have

$$\begin{matrix} a_{11}u_1 + \ldots + a_{1r}u_r = 0, \\ \ldots\ldots\ldots\ldots\ldots \\ a_{n1}u_1 + \ldots + a_{nr}u_r = 0, \end{matrix} \qquad \text{and} \qquad \det \begin{bmatrix} a_{11} \ldots a_{1n} \\ \ldots\ldots \\ a_{n1} \ldots a_{nn} \end{bmatrix} = \pm 1$$

by non-singularity of X. Multiplying this system of equations by $p_\theta = u_2 u_3 \ldots u_s$ and resolving it, we get

$$p_\sigma = u_1 u_2 \ldots u_s = \sum_{l=n+1}^{r} c_l u_l u_2 u_3 \ldots u_s,$$

where c_l are integers. The element $u_l u_2 \ldots u_s$ is null if $a_l, a_2, \ldots, a_s$ span no face in S (here $l > s$). In other case let us denote this face by σ_l. Then $u_l u_2 \ldots u_s = p_{\sigma_l}$ and the assertion of the lemma is satisfied by the elements σ_l.

The monomial $U_1^{r_1} U_2^{r_2} \ldots U_s^{r_s}$ is called *radical* if $0 \leqslant r_1, r_2, \ldots, r_s \leqslant 1$. The coset of such a monomial will be called a *radical element*. Then an element of $Z[U]/I$ is radical iff it is of the form p_σ for some $\sigma \in S$.

LEMMA 2. *The Z-module $Z[U]/I$ is spanned by radical elements.*

Proof. For $q = 0, 1, 2, \ldots$, let $(Z[U]/I)_q$ denote the submodule of q-th degree. Suppose that for some q the module $(Z[U]/I)_{q-1}$ is spanned by radical elements. Then the module $(Z[U]/I)_q$ is spanned by elements of the form $u_j w$ $(j = 1, 2, \ldots, r)$, where $w \in (Z[U]/I)_{q-1}$ is radical, say $w = p_\sigma$ for $\sigma \in S$. If $a_j \,\text{non-}\!\prec \sigma$, then $u_j w$ is radical. In other case, Lemma 1 can be applied to the faces $\{0\} \prec a_j \prec \sigma$. In other words, a_j can be shifted away from σ.

We have

$$u_j = \sum_l c_l u_l,$$

where integers l are such that $a_l \,\text{non-}\!\prec \sigma$. The elements $u_l w$ are either null or radical. Thus the assertion of the lemma holds for $q+1$.

Definition. Let $\omega \in S$, $\dim\omega = n$, and $\sigma \prec \omega$. Let us set

$$C_{\sigma,\omega} = \{\tau \in S\colon \sigma \prec \tau \prec \omega\}.$$

By *cells* in S we mean the subsets of S of the form $C_{\sigma,\omega}$ for some σ and ω. A *filtration* in S is a sequence of subcomplexes

$$(F)\colon S = S_m \supset S_{m-1} \supset \ldots \supset S_0 = \emptyset$$

such that

(1) S_j are closed, i.e. $(\sigma \in S_j,\ \tau \prec \sigma) \Rightarrow (\tau \in S_j)$,

(2) for each j, $S_{j+1} - S_j$ is a cell.

Let a cellular decomposition of S be given, that is to say a set $\{C_1, C_2, \ldots, C_m\}$ of disjoint cells such that $S = \bigcup C_j$. The decomposition is said to be *filtrable* if there exists a filtration (F) of S such that $S_{j+1} - S_j$ is a cell of that decomposition for all j.

PROPOSITION 1. *Let a filtration* (F) *of* S *be given and let* $S_j - S_{j-1} = C_{\sigma_j,\omega_j}$, *where* $\dim\omega_j = n$ *and* $\sigma_j \prec \omega_j$ *for* $j = 1, 2, \ldots, m$. *Then the ring* $Z[U]/I$, *attached to* S *by* (*), *is generated, as a Z-module, by* $p_{\sigma_1}, p_{\sigma_2}, \ldots$ $\ldots, p_{\sigma_m}$ (in fact, those elements form a basis of $Z[U]/I$, as it will follow from the proof of the Theorem in Section 2).

Proof. By Lemma 2 it suffices to show that for each $\sigma \in S$ the element p_σ is of the form $\sum_k d_k p_{\sigma_k}$, where d_k are integers. Suppose that this

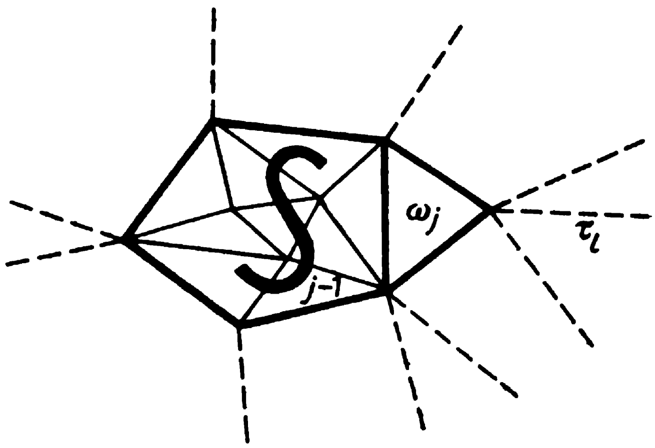

Fig. 1. The spherical complex corresponding to the 3-dimensional conical complex S_j

conclusion holds for all $\sigma \in S - S_j$ for some fixed j. We prove that it also does for $\sigma \in S - S_{j-1}$. Let $\sigma \in S_j - S_{j-1}$. Then, applying Lemma 1 to the triple $\sigma_j \prec \sigma \prec \omega_j$, we have

$$p_\sigma = \sum c_l p_{\tau_l},$$

where $\sigma_j \prec \tau_l$, and hence $\tau_l \notin S_{j-1}$. Otherwise, we have τ_l non-$\prec$ ω_j, and hence $\tau_l \notin S_j - S_{j-1}$ (see Fig. 1) for all l. Then $\tau_l \notin S_j$ and p_{τ_l} is a suitable combination for all l. Then so is p_σ and the proof follows by the "decreasing induction" on j.

Remark. We compare cellular decompositions of S and those of X_S. Since

$$X_S = \bigcup_{\sigma\in S} O^\sigma,$$

for an arbitrary subset $S' \subset S$ we can put

$$X_{S'} = \bigcup_{\sigma\in S'} O$$

(warning: the affine torus embedding X_σ corresponding to a cone σ is not equal to $X_{\{\sigma\}}$). Those are locally closed T-invariant subschemes of X. In particular, let $C = C_{\sigma,\omega}$ be a cell in S. It follows easily from the non-singularity of X that X_C is an affine space. Then each cellular decomposition of S corresponds to a cellular decomposition of X.

A cellular decomposition of X,

$$X = \bigcup_{i=1}^{n} W_i,$$

is said to be *filtrable* if there exists a sequence

$$(\tilde{F})\colon\ X = X_0 \supset X_1 \supset \ldots \supset X_m = \emptyset$$

of closed subschemes of X such that $X_j - X_{j+1}$ is a cell of the decomposition for $j = 0, 1, \ldots, m-1$ (cf. Definition 2 of [2]). Clearly, a decomposition of S is filtrable iff the corresponding decomposition of $X = X_S$ is filtrable. More precisely, to a filtration (F) of S there corresponds a filtration $(\tilde{F})$ of X such that

$$X_j = X_{(S-S_j)} = X - X_{S_j},$$

and then

$$X_j - X_{j+1} = X_{(S_{j+1}-S_j)}.$$

2. Chow ring of a projective toroidal embedding. Let us study the relation between the ring $Z[U]/I$ attached to S and the ring $C^*_{\mathrm{rat}}(X_S)$ of classes of cycles on X_S, i.e. the Chow ring of X. To each face $\sigma \in S$ the orbit $O^\omega \subset X$ corresponds (see [7]). Its closure is a cycle, and we set

$$\mathrm{cl}(\omega) := \mathrm{cl}_{\mathrm{rat}}(\overline{O^\omega}) \in C^*_{\mathrm{rat}}(X).$$

The invariant divisors $\bar{O}^{a_j}$ will be denoted by D_j for $j = 1, 2, \ldots, r$. Consider the homomorphism

$$\chi\colon Z[U_1, U_2, \ldots, U_r] \to C^*_{\mathrm{rat}}(X), \qquad U_j \mapsto \mathrm{cl}(a_j) = \mathrm{cl}(D_j).$$

The linear generators of the ideal I are in $\ker\chi$. Indeed, let us consider the form $(c_1, c_2, \ldots, c_n) \mapsto c_i$ on Z^n (this last group is identified with $\mathrm{Hom}(G_m, T)$). Let $\mathfrak{X}_i$ denote the character on T corresponding to this form (see [7]). For the rational function $\mathscr{X}_i$ on X and the subvariety D_j,

$$\mathrm{val}_{D_j}(\mathscr{X}_i) = a_{ij}.$$

Hence $\sum_j a_{ij}D_j$ is a principal divisor for $i = 1, 2, \ldots, n$.

Clearly, monomial generators of I are also in $\ker\chi$, and hence χ can be factorized by a homomorphism h defined on $Z[U]/I$.

Let l be a number prime to $\mathrm{char}(k)$, and $H^*(X, Q_l)$ the algebra of l-adic cohomology. Consider the following diagram (see [8], the symbol X is omitted):

$$\text{(D)}\quad \begin{tikzcd} Z[U]/I \arrow[r, "h"] \arrow[d, "\iota"] & C^*_{\mathrm{rat}} \arrow[r, "g"] \arrow[d] & C^*_{\mathrm{alg}} \arrow[r, "f"] \arrow[d] & C^*_\tau \arrow[dr] & \\ (Z[U]/I)\otimes_Z Q_l \arrow[r, "h'"] & C^*_{\mathrm{rat}}\otimes Q_l \arrow[r, "g'"] & C^*_{\mathrm{alg}}\otimes Q_l \arrow[r, "f'"] & C^*_\tau\otimes Q_l \arrow[r, "e"] & H^*(X, Q_l) \end{tikzcd}$$

The homomorphism e doubles the gradation and all remaining homomorphisms of the diagram are compatible with the gradation.

THEOREM. *Let X be a non-singular projective torus embedding, S the corresponding conical complex in R^n and $Z[U]/I$ the graded ring* (*) *attached to S. Then the diagram* (D) *induces isomorphisms*

$$Z[U]/I \xrightarrow{\sim} C^*_{\mathrm{rat}}(X) \xrightarrow{\sim} C^*_{\mathrm{alg}}(X),$$
$$Z[U]/I \otimes_Z Q_l \xrightarrow{\sim} H^*(X, Q_l).$$

For each q the Z-module $C^q_{\mathrm{rat}}(X)$ is free. For i odd we have $H^i(X, Q_l) = 0$.

Proof. Using Theorem 2 of [3] we shall show that S has a filtration. Let us choose an element

$$a \in Z^n - \bigcup_{\substack{\sigma\in S\\ \dim\sigma<n}} \sigma$$

and consider the action of G_m on X given by the homomorphism $\lambda_a: G_m \to T$. Fixed points $x_1, x_2, \ldots, x_m$ with respect to this action are the same as fixed points by T, so they correspond to n-dimensional faces $\omega_1, \omega_2, \ldots, \omega_m$ belonging to S. Consider the unstable decomposition of X (see [1] and [2]),

$$X = \bigcup_i W^u(x_i),$$

where $W^u(x_i)$ are affine spaces (1).

It corresponds to the cellular decomposition of S (see the Remark) of the form

$$S = \bigcup_i C_{\sigma_i, \omega_i}$$

for some faces σ_i of ω_i (see [6] for details). By Theorem 2 of [3] the first decomposition is filtrable, then so is the second one.

(1) The sets $W^u(x_i)$ coincide with the cells defined by Ehlers in [4].

Let us fix a filtration $(\tilde{F})$ of X (and the corresponding filtration (F) of S). We begin with proving that the homomorphism h of the diagram (D) is surjective. By Proposition 7 of [5] (p. 4-31), applied to the filtration (F), the Z-module $C^*_{\mathrm{rat}}(X)$ is spanned by classes of cycles $\bar{X}_{C_{\sigma_i,\omega_i}}$, $i = 1, 2, \ldots, m$. Those cycles are nothing else but $\bigcap_{j;a_j \prec \sigma_i} D_j$. Moreover, the divisors D_j intersect transversally. Indeed, by non-singularity of X, those divisors are locally of the form

$$D_j \xrightarrow{\sim} k^1 \times k^1 \times \ldots \times 0_j \times \ldots \times k^1.$$

Then

$$\mathrm{cl}(\bar{X}_{C_{\sigma_i,\omega_i}}) = \prod_{j;a_j \prec \sigma_i} \mathrm{cl} D_j = \prod_{j;a_j \prec \sigma_i} h(u_j) = h(p_{\sigma_i}),$$

and hence $C^*_{\mathrm{rat}}(X)$ is generated by $h(p_{\sigma_i})$, $i = 1, 2, \ldots, m$.

Since X has a cellular decomposition, by [8] the homomorphism e of (D) is an isomorphism. Hence all horizontal arrows in (D) are surjective. It follows from Proposition 1, applied to the considered filtration of S, that the Q_l-space $(Z[U]/I) \otimes_Z Q_l$ is spanned by $p_{\sigma_1} \otimes 1, p_{\sigma_2} \otimes 1, \ldots, p_{\sigma_m} \otimes 1$, and m is the number of cells of a decomposition of S. On the other hand, $\dim H^*(X, Q_l)$ is equal to the number of cells of a decomposition of X (it follows, e.g., from [2]), and hence equal to m. It is also equal to the Euler-Poincaré characteristic of X, since the cohomology in odd dimensions is null. It follows that the composed epimorphism of linear spaces

$$(Z[U]/I) \otimes Q_l \to H^*(X, Q_l)$$

is bijective. Since the elements $p_{\omega_i} \otimes 1$ form a basis of $(Z[U]/I) \otimes Q_l$, p_{ω_i} are independent, and hence they form a basis of the Z-module $Z[U]/I$.

Thus f', g', h' are bijective, t is injective, and hence f, g, h are bijective.

3. Non-projective case. Let now S be such a conical complex that X is non-singular and complete, not necessarily projective. If S has any filtration, then the assertion of the Theorem remains true for X. Unfortunately, the decompositions of X, resulting from the G_m-actions described above, need not to be filtrable (see the Example in [6]; besides, there exists a filtrable decomposition for this example). We are not able to prove that every "non-singular complete" complex S has a filtration (**P 1183**). Nevertheless, we have a provisional

PROPOSITION 2. *If* $\dim S \leqslant 3$, *then* S *has a filtration*

The proof is immediate for dimensions 1 and 2. Instead of a 3-dimensional conical complex in R^3 we consider the corresponding 2-dimensional spherical complex $\dot{E}$, in other words — a triangulation of the 2-dimensional sphere. One should show that there exists a closed filtration

$$E = E_m \supset E_{m-1} \supset \ldots \supset E_0 = \emptyset$$

such that, for $i = 0, 1, \ldots, m-1$, $E_{i+1} - E_i$ has one of the following four forms (see Fig. 2):

(I) $\{\omega\} = C_{\omega,\omega}$,

(II) $\{\omega, \varrho\} = C_{\varrho,\omega}$,

(III) $\{\omega, \varrho, \tau, \alpha\} = C_{\alpha,\omega}$,

(IV) $\{\omega, \varrho, \sigma, \tau, \alpha, \beta, \gamma\} = C_{0,\omega}$

(0 is the origin, hence it has no image in the figure).

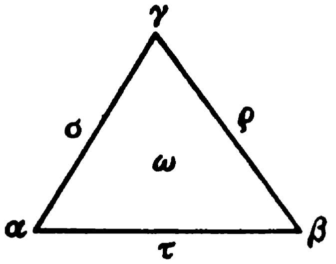

Fig. 2

Moreover, $E_m - E_{m-1}$ has to be of the form (I), $E_1 - E_0$ — of the form (IV), and the remaining $S_{i+1} - S_i$ — of the form (II) or (III). We choose an arbitrary simplex $\omega \in Q$ of dimension 2 and set

$$E_{m-1} = E - \{\omega\}.$$

E_{m-1} may be regarded as a triangulation of a plane set, homeomorphic to the closed disc D of dimension 2. It remains to find a filtration for E_{m-1}. Such a filtration can easily be constructed by induction, with the help of the following

LEMMA 3. *Given a curvilinear triangulation Q of D, containing $m \geqslant 2$ simplices of dimension 2, there exist at least 2 closed subcomplexes Q' and Q'' of Q such that $Q - Q'$ and $Q - Q''$ are of the form* (II) *or* (III) *and both topological spaces $|Q'|$ and $|Q''|$ are homeomorphic to D.*

Proof follows by induction on m. The assertion is obvious for $m = 2$. Suppose that it is true for all m less than some k and consider a triangulation

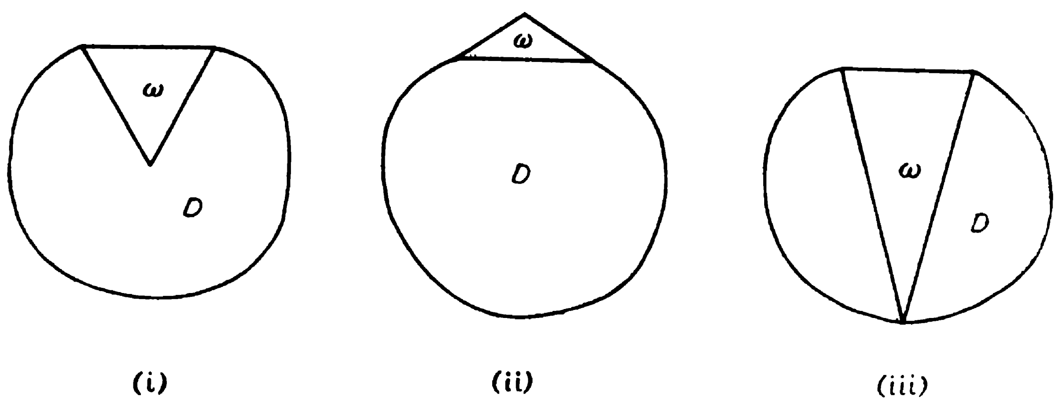

Fig. 3

of D containing k triangles. Let $\omega \in Q$ be such a triangle that at least one edge of ω is contained in the boundary of D. There are three possibilities (Fig. 3).

It suffices to consider the case of triangulation Q containing a triangle ω in position (iii). Removing the closed triangle ω from D, we have a decomposition into connected components:

$$D - |\overline{\omega}| = U_1 \cup U_2.$$

Let $\overline{U}_i$ denote the closure of U_i in D, and P_i the restriction of the triangulation Q to $\overline{U}_i$ for $i = 1, 2$. The triangulation P_i either contains only one triangle or satisfies the assumption of the lemma with $m < k$. In both cases there exists at least one closed subcomplex P_i' in P_i such that

$P_i - P_i'$ is of the form (II) or (III);

$|P_i'|$ is homeomorphic to D;

no edge of ω belongs to $P_i - P_i'$.

In order the last condition to be satisfied we have to prove the existence of two subcomplexes in Q.

Indeed, the subcomplexes

$$Q' = Q - (P_1 - P_1') \quad \text{and} \quad Q'' = Q - (P_2 - P_2')$$

of Q satisfy the assertion of the lemma. Thus Proposition 2 is proved.

COROLLARY. *The Theorem of Section* 2 *holds if the assumption of projectivity of X is replaced by*: "*X is complete of dimension less than or equal to* 3".

4. Ehlers' formula for Betti numbers of X_S.

PROPOSITION 3. *Let S be such that $X = X_S$ is a non-singular complete variety, and let l be an integer prime to* char(k). *Let $b_i = \dim H^i(X, Q_l)$. Then $b_i = 0$ for i even.*

Let P_X be the Poincaré polynomial of X such that

$$P_X(U) = \sum_i b_{2i} U^i,$$

let d_i be the number of faces of codimension i in S, and D_S the polynomial

$$D_S(U) = \sum_i d_i U^i.$$

Then $P_X(U) = D_S(U-1)$, and

$$b_{2i} = \sum_{j=1}^{n} (-1)^{j-i} \binom{j}{i} d_j.$$

This formula was obtained by Ehlers in [4]. We shall show how it follows from the cellular decomposition of X.

Let us consider a cellular decomposition

$$S = \bigcup_j C_{\sigma_j, \omega_j}$$

and the corresponding decomposition

$$X = \bigcup_j W_j$$

(indeed, they exist; for example, this can be the unstable decomposition described in the proof of the Theorem in Section 2). By [2], we have

$$b_{2i} = \#\{j: \dim W_j = i\}.$$

Since the subscheme W_j corresponds to C_{σ_j, ω_j} and $\dim W_j = \operatorname{codim} \sigma_j$, Betti numbers of X can be expressed by

$$b_{2i} = \#\{j: \operatorname{codim} \sigma_j = i\}.$$

Hence both polynomials D_S and P_X are given in terms of the complex S.

Now we present one of the ways of deriving the relation between the both polynomials, which gives the interpretation of D_S.

Consider the complex vector space $N \otimes_Z C$ (where N is the group of one-parameter subgroups of the torus T) and its standard immersion into complex projective n-dimensional space P. The Chow ring of P is $Z[U]/U^{n+1}$, the coset u of U being the class of a hyperplane. Let us denote by $\mathrm{Cl}(\lambda)$, where λ is a subset of P, the class of the cycle $\bar{\lambda}$ (Zariski's closure) in the Chow ring of P. If S' is a subset of the complex S, we write

$$\mathrm{Cl}(S') = \sum_{\omega \in S'} \mathrm{Cl}(\omega).$$

With this notation we can describe the polynomial D_S as

$$D_S(u) = \mathrm{Cl}(S).$$

This equality determines $D_S(u)$ uniquely, since $\deg D_S = n$.

We now find the above-mentioned class using the decomposition of S. Let us begin with a single cell $C_{\sigma,\omega} \subset S$ and let r be the codimension of σ. The cone ω has n faces of dimension $n-1$. Let $V_1, V_2, \ldots, V_n$ be the linear hulls of those faces in N_R, and H_i the open half-space spanned by V_i and ω for $i = 1, 2, \ldots, n$. Then

$$\omega = \bigcap_{i=1}^{n} (V_i \cup H_i).$$

Since σ is a face of ω, we can assume that

$$\sigma = \Big(\bigcap_{i=1}^{r} V_i\Big) \cap \omega.$$

The (not necessarily compact) polyhedron $|C_{\sigma,\omega}|$ the faces of which are precisely the elements of $C_{\sigma,\omega}$ is of the form

$$|C_{\sigma,\omega}| = \bigcap_{i=1}^{r} (V_i \cup H_i) \cap \bigcap_{i=r+1}^{n} H_i.$$

Since $\mathrm{Cl}(V_i) = u$ and $\mathrm{Cl}(H_i) = 1$, it follows easily from the transversality of V_i's that

$$\mathrm{Cl}(C_{\sigma,\omega}) = \prod_{i=1}^{r}\big(\mathrm{Cl}(V_i)+\mathrm{Cl}(H_i)\big)\prod_{i=r+1}^{n}\mathrm{Cl}(H_i) = (u+1)^r.$$

For the whole complex S we have

$$\mathrm{Cl}(S) = \sum_j \mathrm{Cl}(C_{\sigma_j,\omega_j}) = \sum_j (u+1)^{\operatorname{codim}\sigma_j} = \sum_{i=1}^{n} b_{2i}(u+1)^i = P_X(u+1),$$

and the formula is proved.

Added in proof. When the paper was in print, Danilov [9] proved by other methods that the Theorem holds for an arbitrary non-singular complete torus embedding.

REFERENCES

[1] A. Białynicki-Birula, *Some theorems on actions of algebraic groups*, Annals of Mathematics 98 (1973), p. 480-497.

[2] — *On fixed points of torus action on projective varieties*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 22 (1974), p. 1097-1101.

[3] — *Some properties of the decompositions of algebraic varieties determined by actions of a torus*, ibidem 24 (1976), p. 667-674.

[4] F. Ehlers, *Eine Klasse komplexer Mannigfaltigkeiten und die Auflösung einiger isolierter Singularitäten*, Mathematische Annalen 218 (1975), p. 127-156.

[5] A. Grothendieck, *Sur quelques propriétés fondamentales en théorie des intersections*, Séminaire Chevalley 1958, p. 4-01 - 4-36.

[6] J. Jurkiewicz, *An example of algebraic torus action which determines the non-filtrable decomposition*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 25 (1977), p. 1089-1092.

[7] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics 339 (1973).

[8] S. L. Kleiman, *Algebraic cycles and the Weil conjectures*, p. 359-386 in: Dix exposés sur la cohomologie des schémas, Amsterdam 1968.

[9] В. И. Данилов, *Геометрия торических многообразий*, Успехи математических наук 33 (2) (1978), p. 85-134.

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