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ON THE FUNCTIONS $\varphi(n)$ *AND* $\sigma(n)$

BY

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In this paper $\varphi(n)$ and $\sigma(n)$ denote the Euler's function and the sum of the divisors of n, respectively, p denotes odd primes, p_i the i-th prime.

It has been asked in [5] whether the inequality

$$\liminf \frac{\overbrace{\sigma \dots \sigma}^{k \text{ times}}(n)}{n} < \infty$$

holds for every k and it has been remarked that for $k = 2$ the affirmative answer follows from a certain deep theorem of Rényi [4]. The aim of this paper is to give an elementary proof of the equality

$$\liminf \frac{\sigma\sigma(n)}{n} = 1$$

and to evaluate other similar limits.

THEOREM. *The following formulae hold:*

$$\liminf \frac{\sigma\sigma(n)}{n} = 1, \tag{1}$$

$$\limsup \frac{\varphi\sigma(n)}{n} = \infty, \tag{2}$$

$$\limsup \frac{\varphi\varphi(n)}{n} = \tfrac{1}{2}, \tag{3}$$

$$\liminf \frac{\sigma\varphi(n)}{n} \leqslant \inf_{4|m} \frac{\sigma\varphi(m)}{m} \leqslant \tfrac{1}{2} + \frac{1}{2^{34}-4}. \tag{4}$$

The proof is based on two lemmata. The first is a generalization of a result of Bojanić [2] and is elementary, the second is related to a theorem of Rényi and is used only to show (3) and (4).

LEMMA 1. *If a is an integer >1 and $N(a,p) = (a^p-1)/a-1$, then*

$$\lim_{p\to\infty} \frac{\varphi(N(a,p))}{N(a,p)} = \lim_{p\to\infty} \frac{\sigma(N(a,p))}{N(a,p)} = 1.$$

Proof. Put $N(a,p) = N = q_1^{a_1} q_2^{a_2} \ldots q_s^{a_s}$, where q_i $(1 \leqslant i \leqslant s)$ are different primes, $a_i \geqslant 1$. Clearly

$$\prod_{i=1}^{s} \left(1-\frac{1}{q_i}\right)^{-1} \geqslant \frac{\sigma(N)}{N} \geqslant 1 \geqslant \frac{\varphi(N)}{N} \geqslant \prod_{i=1}^{s} \left(1-\frac{1}{q_i}\right). \tag{5}$$

For $p > a+1$, we have $p \nmid a-1$, thus $q_i \equiv 1 \pmod{p}$ (cf. [3], p. 381) and $q_i > p$ $(i = 1, 2, \ldots, s)$. It follows that

$$N \geqslant p^{a_1+\ldots+a_s} \geqslant p^s$$

and

$$s \leqslant \frac{\log N}{\log p} = \frac{\log\left(\dfrac{a^p-1}{a-1}\right)}{\log p} < \frac{\log a^p}{\log p} = \frac{p\log a}{\log p}.$$

Hence

$$\prod_{i=1}^{s} \left(1-\frac{1}{q_i}\right) \geqslant \left(1-\frac{1}{p}\right)^s > \left(1-\frac{1}{p}\right)^{p\frac{\log a}{\log p}} \to 1. \tag{6}$$

The lemma follows from (5) and (6).

LEMMA 2. *The following formula holds:*

$$\limsup \frac{\varphi\left(\frac{1}{2}(p-1)\right)}{\frac{1}{2}(p-1)} = \liminf \frac{\sigma\left(\frac{1}{2}(p-1)\right)}{\frac{1}{2}(p-1)} = 1.$$

Proof. Clearly

$$\frac{\varphi\left(\frac{1}{2}(p-1)\right)}{\frac{1}{2}(p-1)} \leqslant 1 \leqslant \frac{\sigma\left(\frac{1}{2}(p-1)\right)}{\frac{1}{2}(p-1)}. \tag{7}$$

On the other hand, it has been proved by Wang ([6], Appendix, formulae (7) and (8)) that

$$P_\omega(x, q, x^{1/6,5\eta}) > 12{,}9\eta \frac{c_q x}{\varphi(q)\log^2 x} + O\left(\frac{c_q x}{\log^3 x}\right).$$

Here, $P_\omega(x, q, \xi)$ is the number of primes p satisfying $p \leqslant x, p \equiv a \pmod{q}$, $p \not\equiv a_i \pmod{p_i'}$ $(i = 1, \ldots, r)$, where $\omega = \langle a, q, a_i\ (1 \leqslant i \leqslant r)\rangle$ is a sequence of integers such that $q \leqslant x$, $(a, q) = 1$, $a_i \not\equiv 0 \pmod{p_i'}$

and p_i' are all primes $\leqslant \xi$ not dividing $2q$; c_q is a certain positive constant (cf. [6], formula (6)), $\eta = \delta/(\delta-1)$, where as stated on p. 1054 one can take $\delta = 1{,}5$. It follows after the substitution $\omega = \langle 3, 4, \overbrace{1, \ldots, 1}^{r \text{ times}}\rangle$ that there exist infinitely many primes p such that every prime factor of $(p-1)/2$ is greater than $p^{1/20}$. Let ε be any number >0 and take p of the above kind greater than $20^{20}\varepsilon^{-20}$. Let

$$\tfrac{1}{2}(p-1) = q_1^{a_1} q_2^{a_2} \ldots q_s^{a_s},$$

where q_i $(1 \leqslant i \leqslant s)$ are different primes and $a_i \geqslant 1$. Clearly $s < 20$, and

$$\prod_{i=1}^{s}\left(1-\frac{1}{q_i}\right) > \left(1-\frac{1}{p^{1/20}}\right)^{20} > 1-\frac{20}{p^{1/20}} > 1-\varepsilon.$$

On the other hand,

$$\prod_{i=1}^{s}\left(1-\frac{1}{q_i}\right)^{-1} \geqslant \frac{\sigma\big(\frac{1}{2}(p-1)\big)}{\frac{1}{2}(p-1)} \geqslant \frac{\varphi\big(\frac{1}{2}(p-1)\big)}{\frac{1}{2}(p-1)} \geqslant \prod_{i=1}^{s}\left(1-\frac{1}{q_i}\right).$$

It follows that

$$(1-\varepsilon)^{-1} > \frac{\sigma\big(\frac{1}{2}(p-1)\big)}{\frac{1}{2}(p-1)} \geqslant \frac{\varphi\big(\frac{1}{2}(p-1)\big)}{\frac{1}{2}(p-1)} > 1-\varepsilon.$$

In view of (7), this completes the proof.

Proof of the Theorem. We begin with formula (1). For any $\varepsilon > 0$ we take a prime $r > 1+\varepsilon^{-1}$ and put $a = r$ in Lemma 1. We have $N(r, p) = \sigma(r^{p-1})$. Hence

$$\lim_{p\to\infty} \frac{\sigma\sigma(r^{p-1})}{r^{p-1}} = \lim_{p\to\infty} \frac{\sigma\sigma(r^{p-1})}{\sigma(r^{p-1})} \cdot \frac{\sigma(r^{p-1})}{r^{p-1}}$$

$$= \lim_{p\to\infty} \frac{\sigma\big(N(r,p)\big)}{N(r,p)} \cdot \lim_{p\to\infty} \frac{\sigma(r^{p-1})}{r^{p-1}} = \frac{r}{r-1} < 1+\varepsilon.$$

Since $\sigma\sigma(n)/n \geqslant 1$ for all n, formula (1) is proved.

Proof of formula (2) is similar. For any M we take a number t such that

$$\prod_{i=1}^{t} \frac{p_i}{p_i-1} > M$$

and put successively $a = p_1, p_2, \ldots, p_t$ in Lemma 1. We have

$$\sigma\Big(\prod_{i=1}^{t} p_i^{p-1}\Big) = \prod_{i=1}^{t} N(p_i, p).$$

Hence

$$\limsup_{p\to\infty} \frac{\varphi\sigma\Big(\prod\limits_{i=1}^{t} p_i^{p-1}\Big)}{\prod\limits_{i=1}^{t} p_i^{p-1}} = \limsup_{p\to\infty} \frac{\varphi\Big(\prod\limits_{i=1}^{t} N(p_i, p)\Big)}{\prod\limits_{i=1}^{t} p_i^{p-1}} \geqslant \limsup_{p\to\infty} \prod_{i=1}^{t} \frac{\varphi\big(N(p_i, p)\big)}{p_i^{p-1}}$$

$$= \prod_{i=1}^{t} \lim_{p\to\infty} \frac{\varphi\big(N(p_i, p)\big)}{N(p_i, p)} \cdot \prod_{i=1}^{t} \lim_{p\to\infty} \frac{N(p_i, p)}{p_i^{p-1}} = \prod_{i=1}^{t} \frac{p_i}{p_i - 1} > M.$$

This completes the proof of (2).

Formula (3) follows at once from Lemma 2, since

$$\limsup_{p\to\infty} \frac{\varphi\varphi(p)}{p} = \limsup_{p\to\infty} \frac{\varphi(p-1)}{p} \geqslant \limsup_{p} \frac{\varphi\big(\frac{1}{2}(p-1)\big)}{\frac{1}{2}(p-1)} \cdot \frac{p-1}{2p},$$

and, on the other hand, $\varphi\varphi(n)/n \leqslant \frac{1}{2}$ for all $n > 1$.

In order to prove formula (4) assume that m is any positive integer divisible by 4. By Lemma 2

$$\liminf_{p\to\infty} \frac{\sigma\varphi(\frac{1}{2}mp)}{\frac{1}{2}mp} \leqslant \liminf_{p\to\infty} \frac{\sigma\big(2\varphi(\frac{1}{2}m)\big)\sigma\big(\frac{1}{2}(p-1)\big)}{\frac{1}{2}mp}$$

$$= \frac{\sigma\varphi(m)}{m} \cdot \liminf_{p\to\infty} \frac{\sigma\big(\frac{1}{2}(p-1)\big)}{\frac{1}{2}p} = \frac{\sigma\varphi(m)}{m}.$$

Since

$$\frac{\sigma\varphi(2^{34}-4)}{2^{34}-4} = \frac{2^{33}-1}{2^{34}-4} = \frac{1}{2} + \frac{1}{2^{34}-4},$$

the proof of the theorem is complete.

The following equalities supplement the theorem:

$$\limsup \frac{\sigma\sigma(n)}{n} = \infty, \tag{8}$$

$$\liminf \frac{\varphi\sigma(n)}{n} = 0, \tag{9}$$

$$\liminf \frac{\varphi\varphi(n)}{n} = 0, \tag{10}$$

$$\limsup \frac{\sigma\varphi(n)}{n} = \infty. \tag{11}$$

Equalities (8) and (10) are trivial, equalities (9) and (11) have been proved by Alaoglu and Erdős [1]. In that paper the following conjecture has been announced: for sufficiently large n the sequence

$$\sigma(n),\ \sigma\sigma(n),\ \varphi\sigma\sigma(n),\ \sigma\varphi\sigma\sigma(n),\ \ldots$$

tends to infinity. We remark that this conjecture implies the finiteness of the set of Mersenne primes. Indeed, if 2^p-1 is a prime, then

$$\varphi\sigma\sigma(2^{p-1}) = \varphi\sigma(2^p-1) = \varphi(2^p) = 2^{p-1},$$

and the sequence in question is periodical.

It seems a natural question to ask whether formula (4) can be improved. Mrs. K. Kuhn has investigated the quotient $\sigma\varphi(n)/n$ for n having at most 6 prime factors and has found that $\sigma\varphi(n)/n \geqslant \frac{1}{2}$ for such n's, the equality being realized only if $n = 2^{2^i+1}-2$ $(0 \leqslant i \leqslant 5)$. This suggests a problem

P 486. Is the inequality $\sigma\varphi(n)/n \geqslant \frac{1}{2}$ true for all n?

Remark. Even the weaker inequality $\inf \sigma\varphi(n)/n > 0$ remains still unproved.

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